

HOLOMORPHIC SPINOR OBSERVABLES IN THE CRITICAL ISING MODEL

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ABSTRACT. We introduce a new version of discrete holomorphic observables for the critical planar Ising model. These observables are holomorphic spinors defined on double covers of the original multiply connected domain. We compute their scaling limits, and show their relation to the ratios of spin correlations, thus providing a rigorous proof to a number of formulae for those ratios predicted by CFT arguments.

1. INTRODUCTION

1.1. The critical Ising model and Smirnov's holomorphic observables.

The two-dimensional Ising model is one of the most well-studied models in statistical mechanics. Given a discrete planar domain Ω^δ (a bounded subset of the square grid), the Ising model in Ω^δ can be viewed either as a random assignment of spins to the *faces* of Ω^δ , or a random collection of edges of Ω^δ , with an edge drawn between each pair of faces having different spins. The partition function of the model is given by

$$\sum_{\sigma: \mathcal{F}(\Omega^\delta) \rightarrow \{-1;1\}} \exp \left[\frac{1}{T} \sum_{f \sim f'} \sigma(f) \sigma(f') \right] \quad \text{or} \quad \sum_{S \in \text{Conf}(\Omega^\delta)} x^{|S|},$$

respectively, where $\mathcal{F}(\Omega^\delta)$ denotes the set of faces of Ω^δ and $\text{Conf}(\Omega^\delta)$ is the set of subgraphs S of Ω^δ such that all vertices of Ω^δ have even degrees in S . We refer the reader to Section 2 for a more detailed discussion and notation. We will be interested in the properties of the model at the critical temperature $T = 2 \log^{-1}(\sqrt{2} + 1)$, which corresponds to $x = \sqrt{2} - 1$. This value of x will be fixed throughout the paper.

Discrete holomorphic observables, also called holomorphic fermions or fermionic observables, were proposed by Smirnov in [Smi06] as a tool to study the critical Ising model, although similar objects appeared earlier in [KC71] and [Mer01] without discussing corresponding boundary value problems. Since then, these observables proved to be very useful for a rigorous analysis of the planar Ising model at criticality in the scaling limit when Ω^δ approximates some continuous domain Ω as the lattice mesh δ tends to zero.

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Recall that Smirnov's fermionic observable is defined as

$$F(a, z) := (-in_a)^{-1/2} \sum_{S \in \text{Conf}_{a,z}} x^{|S|} e^{-i\mathbf{w}(\gamma)/2}, \quad (1.1)$$

where $\text{Conf}_{a,z}$ is the set of edge subsets S , such that S can be decomposed into a disjoint collection of loops and a simple lattice path γ connecting a boundary edge a to the midpoint $z = z_e$ of an inner edge e ; $\mathbf{w}(\gamma)$ is the winding number of γ ; and $n_a \in \{\pm 1, \pm i\}$ denotes the orientation of the outgoing boundary edge a . With this definition, the observable has been shown to be discrete holomorphic and satisfy Riemann-type boundary conditions

$$F(a, z) \sqrt{in_z} \in \mathbb{R}, \quad z \in \partial\Omega^\delta \setminus \{a\}. \quad (1.2)$$

This led to a proof of its convergence to a conformally covariant scaling limit [CS12].

This result has been the main ingredient of the recent progress in rigorous understanding of conformal invariance in the critical two-dimensional Ising model. The martingale property of $F(a, z)$ (see further details in [CS12]) allows one to prove convergence of the Ising interfaces to the chordal Schramm's SLE_3 curves. Using a slightly different version of this observable, Hongler and Smirnov [HS10] were able to compute the scaling limit of the energy density in the critical Ising model on the square grid, including the lattice dependent constant before the conformally covariant factor. This result was later extended to all correlations of the energy density field and certain boundary spin correlations [Hon10].

At the same time, similar observables proved to be very useful in the analysis of the random cluster (Fortuin-Kasteleyn) representation of the critical Ising model [Smi06, RC06, Smi10, CS12, DHN11]. In particular, it was shown by Beffara and Duminil-Copin [BD10] that they can be used to give a short proof of criticality of the Ising model at the self-dual point.

Many of these results generalize beyond the case of square grid approximations. Thus, convergence of fermionic observables has been proven for isoradial lattices [CS12], which reappeared in the connection with the critical Ising model in the paper of Mercat [Mer01]. This proved the universality phenomenon, i.e., the fact that a microscopic structure of the lattice does not affect macroscopic properties of the scaling limit. Moreover, discrete complex analysis technique developed in [CS11] and [CS12] provides a general framework for such universal proofs.

On the other hand, one of the most natural questions about the Ising model – the rigorous proof of conformal covariance of spin correlations in the scaling limit – remained out of reach until recently. The goal of the present work is to introduce a new tool – spinor holomorphic observables – that allows to attack this problem. In particular, we prove convergence of ratios of spin correlations corresponding to different boundary conditions to conformally invariant limits. In a subsequent joint paper with Clément Hongler [CHI12], using a more elaborate version of the spinor observables, we prove conformal covariance of spin correlations themselves.

1.2. Spinor holomorphic observables and ratios of spin correlations. In this paper we extend the study of fermionic observables to the case of multiply connected domains. Given a double cover $\varpi : \tilde{\Omega}^\delta \rightarrow \Omega^\delta$ of such a domain, we define the observable $F_\varpi(a, \cdot) : \tilde{\Omega}^\delta \rightarrow \mathbb{C}$ by

$$F_\varpi(a, z) := (-in_a)^{-1/2} \sum_{S \in \text{Conf}_{\varpi(a), \varpi(z)}} x^{|S|} e^{-i\mathbf{w}(\gamma)/2} (-1)^{l(S) + \mathbf{1}_{\gamma \cdot a \rightarrow z}}, \quad (1.3)$$

where $a, z \in \tilde{\Omega}^\delta$, but the sum is taken over *the same set of configurations* as before; $l(s)$ is the number of loops in S that do not lift as closed loops to $\tilde{\Omega}^\delta$, and $\mathbf{1}_{\gamma:a \rightarrow z}$ is the indicator of the event that γ lifts to $\tilde{\Omega}^\delta$ as a path running from a to z (and not to the other sheet), see Section 3 for detailed discussion. In other words, we plug into (1.1) an additional sign that depends on homology class of S modulo two. It is worth to mention that our observables should be closely related to the vector bundle Laplacian technique applied to uniform spanning trees and double dimers by Kenyon [Ken10, Ken11], although at the moment we do not know any exact correspondence of that sort.

Our main observation is that $F_\varpi(a, z)$ are discrete holomorphic and satisfy the boundary conditions (1.2), just like Smirnov's observable $F(a, z)$. The definition implies that $F_\varpi(a, z) = -F_\varpi(a, z^*)$, if $z \neq z^*$ belong to a fiber of the same point; hence, we call F_ϖ *holomorphic spinors*.

To describe the scaling limits of $F_\varpi(a, \cdot)$, we will introduce the continuous holomorphic spinors $f_\varpi(a, \cdot)$. Roughly speaking, these are fundamental solutions to the continuous Riemann boundary value problem (1.2) on the double-cover $\tilde{\Omega}$, with a singularity at a and the property $f_\varpi(z) \equiv -f_\varpi(z^*)$. Postponing precise definitions until Section 3, we now state our first main result (see Theorem 3.13):

Theorem A. *Suppose that Ω^δ is a sequence of discrete domains of mesh size δ approximating (in the sense of Carathéodory) a continuous finitely connected domain Ω , and that $a^\delta \in \partial\Omega^\delta$ tends to $a \in \partial\Omega$ as $\delta \rightarrow 0$. Then there is a sequence of normalizing factors $\beta(\delta) = \beta(\delta; \Omega^\delta, a^\delta, \varpi)$ such that*

$$\beta(\delta)F_\varpi(a^\delta, \cdot) \rightarrow f_\varpi(a, \cdot), \quad \delta \rightarrow 0$$

uniformly on compact subsets of Ω .

This convergence also holds true up to the “nice” parts of the boundary; moreover, considering ratios of observables corresponding to different ϖ 's, one can get rid of normalization issues. We work this out in Theorem 3.16.

A striking feature of our new observables is their direct relation to spin correlations. Let Ω^δ be a *simply connected* domain with m punctures, that is, m single faces f_1, \dots, f_m removed, and let ϖ be the cover that branches around each of these punctures. Then, it turns out that $F_\varpi(a, b)$, $b \in \partial\tilde{\Omega}^\delta$, is (up to a fixed complex factor, see Proposition 3.6) equal to

$$Z_{ab}\mathbb{E}_{ab}[\sigma(f_1) \dots \sigma(f_m)],$$

where Z_{ab} and \mathbb{E}_{ab} stand for the partition function and the expectation for the Ising model with Dobrushin boundary conditions: “−” on the (ab) boundary arc and “+” on (ba) . This, together with convergence results for the observables, gives the following corollary (with the notation “ \mathbb{E}_+ ” referring to “+” boundary conditions everywhere on $\partial\Omega^\delta$):

Corollary B. *Let $(\Omega^\delta, a^\delta, b^\delta)$ approximate (Ω, a, b) as $\delta \rightarrow 0$. Then*

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(z_1^\delta) \dots \sigma(z_m^\delta)]}{\mathbb{E}_+[\sigma(z_1^\delta) \dots \sigma(z_m^\delta)]} \xrightarrow{\delta \rightarrow 0} \vartheta(\phi(z_1), \dots, \phi(z_m)), \quad (1.4)$$

where $\vartheta = \vartheta_{\infty,0}^{\mathbb{C}_+}$ are explicit functions and ϕ is a conformal map from Ω onto the upper half-plane \mathbb{C}_+ sending a to ∞ and b to 0.

In Section 6 we give explicit formulae for ϑ in \mathbb{C}_+ , and hence, by conformal invariance, for all simply connected domains. For example,

$$\vartheta_{ab}^\Omega(z) = \cos[\pi \text{hm}_\Omega(z, (ab))],$$

where $\text{hm}_\Omega(z, (ab))$ stands for the harmonic measure of the arc (ab) in Ω as viewed from z . These formulae for $m = 1, 2$ were previously conjectured by means of Conformal Field Theory, see [BG93] and earlier papers. To the best of our knowledge, the explicit formulae for $m \geq 3$ are new.

Corollary B admits a number of generalizations. Let Ω^δ approximate a finitely connected domain Ω with k inner boundary components $\gamma_1, \dots, \gamma_k$ (possibly macroscopic). Then, for any $m \leq k$, one has

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(\gamma_1^\delta) \dots \sigma(\gamma_m^\delta)]}{\mathbb{E}_+[\sigma(\gamma_1^\delta) \dots \sigma(\gamma_m^\delta)]} \xrightarrow{\delta \rightarrow 0} \vartheta_{ab}^\Omega(\gamma_1, \dots, \gamma_m), \quad (1.5)$$

where the functions $\vartheta_{ab}^\Omega(\gamma_1, \dots, \gamma_m)$ are conformally invariant, the expectations \mathbb{E}_{ab} , \mathbb{E}_+ are taken for the Ising model with Dobrushin (respectively, “+”) boundary conditions on the outer boundary component and monochromatic on inner components γ_j , meaning that we constrain the spins to be the same along each component, but do not specify a priori whether it is plus or minus. In this case, $\sigma(\gamma_j^\delta)$ denotes the (random) spin of the component γ_j .

Further, closely following the route proposed by Hongler in [Hon10], we prove a Pfaffian formula which generalizes (1.5) to the case of $2n$ boundary change operators (in other words, “+/-/-/+/-” boundary conditions with $2n$ marked boundary points, see Section 5). For $m = 1, 2$ this Pfaffian formula (along with the expressions for ϑ) was previously derived by means of Conformal Field Theory [BG93], whereas we give it a rigorous proof for general m both in discrete, and, thanks to the convergence theorem, in continuous settings.

Another application of our new observables [Izy11] is the proof of convergence of (multiple) Ising interfaces to SLE curves in multiply connected domains. In that context, a proper choice of the observable F_ϖ (i.e., the corresponding double cover ϖ) guarantees its martingale property with respect to the growing interface. To prove that property, it is important to relate the values of F_ϖ to the partitions function of the model with relevant boundary conditions. In Section 5, we show how to do it in the most general case, see Proposition 5.6. The simplest example of an SLE process treated in this way (for which the use of a non-trivial double cover is essential) is a radial Ising interface converging to radial SLE₃.

For simplicity, in the present paper we work on the square grid, but all our proofs remain valid for the self-dual Ising model defined on isoradial graphs (e.g., see [CS12]). We refer the reader interested in a detailed presentation of the basic notions of discrete complex analysis on those graphs to the paper [CS11] and the reader interested in the history of the Ising model to the paper [CS12] and references therein.

1.3. Organization of the paper. In Section 2, we fix the notations and conventions regarding discrete domains and the Ising model. In Section 3, we give the definition of the spinor observable and discuss its properties (in particular, discrete holomorphicity and boundary conditions), as well as the connections to spin correlations. We then define the continuous counterparts of the observables and briefly discuss their properties. Section 4 is devoted to the proof of main convergence

results for spinor observables: Theorem 3.13 (convergence in the bulk) and Theorem 3.16 (convergence on the boundary). We generalize our results to the case of multiple boundary change operators in Section 5. Finally, in Section 6 we give explicit formulae for the continuous observables f_ϖ in the punctured half-plane and for the scaling limits $\vartheta_{\infty,0}^{\mathbb{C}+}$ appearing in Corollary B.

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2. NOTATION AND CONVENTIONS

2.1. Graph notation. By *(bounded) discrete domain* (of mesh δ) Ω^δ we mean a (bounded) *connected* subset of the square lattice $\delta\mathbb{Z}^2$ (an example of a discrete domain is given on Fig. 1). More precisely, a discrete domain is specified by three sets $\mathcal{V}(\Omega^\delta)$ (vertices), $\mathcal{F}(\Omega^\delta)$ (faces) and $\mathcal{E}(\Omega^\delta) = \mathcal{E}_{\text{in}}(\Omega^\delta) \cup \mathcal{E}_{\text{bd}}(\Omega^\delta)$ (interior edges and boundary half-edges, respectively), with the following requirements:

- all four edges and vertices incident to any face $f \in \mathcal{F}(\Omega^\delta)$ belong to $\mathcal{E}(\Omega^\delta)$;
- every vertex in $\mathcal{V}(\Omega^\delta)$ is incident to four edges or half-edges in $\mathcal{E}(\Omega^\delta)$;
- every vertex that is incident to at least one edge or half-edge $e \in \mathcal{E}(\Omega^\delta)$ belongs to $\mathcal{V}(\Omega^\delta)$;
- at least one of two faces incident to any edge $e \in \mathcal{E}_{\text{in}}(\Omega^\delta)$ belongs to $\mathcal{F}(\Omega^\delta)$.

For an interior edge $e \in \mathcal{E}_{\text{in}}(\Omega^\delta)$ we denote by z_e its midpoint. For a boundary half-edge $e \in \mathcal{E}_{\text{bd}}(\Omega^\delta)$ we denote by z_e its endpoint which is not a vertex of Ω^δ . When no confusion arises we will identify an edge (or half-edge) e with a point z_e .

By the *boundary* $\partial\Omega^\delta$ of Ω^δ we will mean the set of all its boundary half-edges $\mathcal{E}_{\text{bd}}(\Omega^\delta)$ or, if no confusion arises, the set of corresponding endpoints z_e .

A *double cover* of a discrete domain Ω^δ is a graph $\tilde{\Omega}^\delta$ with a two-to-one local graph isomorphism $\varpi : \tilde{\Omega}^\delta \rightarrow \Omega^\delta$. Given a marked boundary half-edge $a \in \partial\Omega^\delta$, one can describe points z on a double cover by lattice paths γ running from a to z in Ω^δ , modulo homotopy and modulo an appropriate subgroup of the fundamental group. If Ω^δ is $(k+1)$ -connected, that is, has k holes, then there are 2^k double covers, including the trivial one. Namely, to define a cover, for each hole one has to specify whether a loop surrounding this hole lifts to a loop in the double-cover, or to a path connecting points on different sheets. In the latter case we will say that ϖ *branches* around that hole. If z is a point on a double cover $\tilde{\Omega}^\delta$, we let $z^* \in \tilde{\Omega}^\delta$ be defined by $\varpi(z^*) = \varpi(z)$ and $z^* \neq z$. We will also use the obvious notation $\mathcal{V}(\tilde{\Omega}^\delta)$, $\mathcal{E}(\tilde{\Omega}^\delta)$ etc.

2.2. Ising model notation. We will work with the low-temperature contour representation of the critical Ising model in Ω^δ (see [Pal07]). We call a subset S of edges and half-edges in Ω^δ (see Fig. 1, note that we admit inner half-edges in S) a *generalized configuration* or a *generalized interfaces picture* for this model, if

- each vertex in Ω^δ is incident to 0, 2 or 4 edges and half-edges in S ;
- if an edge $e = e' \cup e''$ consists of two halves e', e'' , then at most one of those three belongs to S .

We will denote the set of all generalized configurations in Ω^δ by $\text{Conf}_{\text{gen}}(\Omega^\delta)$. By the *boundary* ∂S of $S \in \text{Conf}_{\text{gen}}(\Omega^\delta)$ we will mean the set of all half-edges $e \in S$ or corresponding points z_e , if no confusion arises. The *partition function* of the critical Ising model is given by

$$Z(\Omega^\delta) = \sum_{S \in \text{Conf}(\Omega^\delta)} x^{|S|}, \quad x = x_{\text{crit}} = \sqrt{2} - 1 \quad (2.1)$$

(the value $x = x_{\text{crit}}$ will be fixed throughout the paper). Here and below $|S|$ is the total number of edges and half-edges in S , and

$$\text{Conf}(\Omega^\delta) := \{S \in \text{Conf}_{\text{gen}}(\Omega^\delta) : \partial S \subset \partial\Omega^\delta\}.$$

The formula (2.1) endows the set $\text{Conf}(\Omega^\delta)$ of configurations, which corresponds to free boundary conditions in the spin representation, with a probability measure, the probability of a particular configuration S being $x^{|S|}/Z(\Omega^\delta)$.

We will mostly work with subsets of $\text{Conf}(\Omega^\delta)$, and restrictions of the probability measure to those subsets. Thus, we denote

$$\begin{aligned} \text{Conf}_+(\Omega^\delta) &:= \{S \in \text{Conf}(\Omega^\delta) : \partial S = \emptyset\}, \\ \text{Conf}_{e_1, \dots, e_n}(\Omega^\delta) &:= \{S \in \text{Conf}(\Omega^\delta) : \partial S = \{z_{e_1}, \dots, z_{e_n}\} \bmod 2\}, \end{aligned} \quad (2.2)$$

where “mod 2” means that if some of e_1, \dots, e_n appear several times in the subscript (it will be useful for us to allow this), we keep in ∂S only those appearing an odd number of times. In the spin representation, the subset $\text{Conf}_+(\Omega^\delta)$ corresponds to *locally monochromatic* boundary conditions, that is, along each boundary component the spins are required to be the same (although they may be different on different components). If $a_1, \dots, a_{2n} \in \partial\Omega^\delta$, then $\text{Conf}_{a_1, \dots, a_{2n}}(\Omega^\delta)$ corresponds to the configurations where the spins change from “+” to “−” at the boundary points (half-edges) a_1, \dots, a_{2n} .

Remark 2.1. To simplify the notation, we will write $\text{Conf}_{e_1, \dots, e_n}(\Omega^\delta)$ instead of $\text{Conf}_{\varpi(e_1), \dots, \varpi(e_n)}(\Omega^\delta)$ when $e_1, \dots, e_n \in \tilde{\Omega}^\delta$. One should remember that we always consider Ising configurations or generalized interfaces pictures in the planar domain Ω^δ itself, even though we will define observables on double covers $\tilde{\Omega}^\delta$.

3. SPINOR HOLOMORPHIC OBSERVABLES AND THEIR LIMITS

3.1. Discrete holomorphic spinor observables. In this subsection we will construct spinor observables and prove their discrete holomorphicity. These observables should be considered as natural generalizations of fermionic observables introduced by Smirnov [Smi06, CS12] to the multiply connected setup. A discrete domain Ω^δ , its double cover $\varpi : \tilde{\Omega}^\delta \rightarrow \Omega^\delta$, and a boundary half-edge $a \in \partial\tilde{\Omega}^\delta$ will be fixed throughout this subsection. In order to give a consistent definition for all discrete

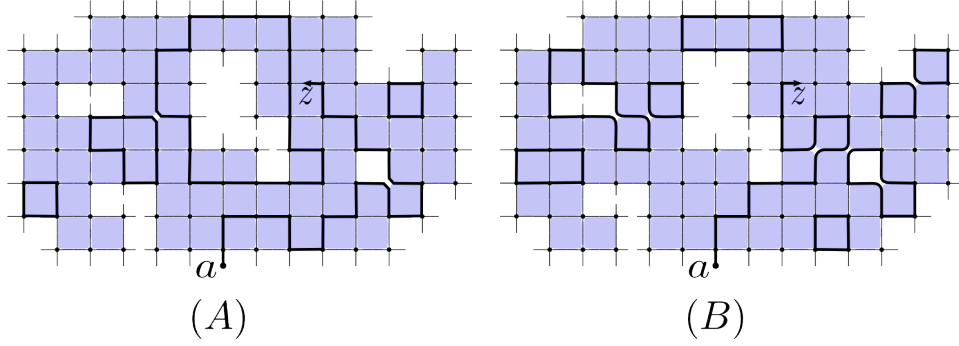


FIGURE 1. An example of four-connected discrete domain Ω^δ and two generalized interfaces pictures $S_{A,B} \in \text{Conf}_{a,z}(\Omega^\delta)$, each decomposed into a collection of loops and a simple lattice path $\gamma_{A,B} : a \rightsquigarrow z$, as required in Definition 3.1. For a double cover $\tilde{\Omega}^\delta$, branching, say, around two small holes but not around the big central one, one has $l(S_A) = 0$, $l(S_B) = 2$, and $s(z, \gamma_A) = -s(z, \gamma_B)$ as, being lifted on $\tilde{\Omega}^\delta$, γ_A and γ_B end on different sheets.

domains, we need the following (technical) notation. The half-edge a , oriented from an inner vertex to z_a , can be thought of as a complex number. Then we set

$$\eta_a := e^{-\frac{i}{2}(\arg(a) + \frac{1}{2}\pi)} = (ia/|a|)^{-\frac{1}{2}} \quad (3.1)$$

for some fixed choice of the sign. Note that, if a is south-directed, then $\eta_a = \pm 1$.

Given a point $z = z_e$ (i.e., the midpoint of an edge or the endpoint of a boundary half-edge $e \in E(\tilde{\Omega}^\delta)$) and a configuration $S \in \text{Conf}_{a,e}(\Omega^\delta)$, we introduce the *complex phase* of S with respect to z . First, we decompose S into a collection of loops and a path γ running from $\varpi(a)$ to $\varpi(e)$ so that there are no transversal intersections or self-intersections (see Fig. 1). A loop in such a decomposition will be called *non-trivial* if it does not lift to a closed loop on the double cover (that is, if it lifts to a path between points on different sheets). We denote by $l(S)$ the number of non-trivial loops in S . We also introduce a sign $s(z, \gamma) := +1$, if γ lifts to a path from a to z on $\tilde{\Omega}^\delta$, and $s(z, \gamma) := -1$, if it lifts to a path from a to z^* .

Definition 3.1. *The complex phase of a configuration $S \in \text{Conf}_{a,z}(\Omega^\delta)$ with respect to a point z lying on a double cover $\varpi : \tilde{\Omega}^\delta \rightarrow \Omega^\delta$ is defined as*

$$W_\varpi(z, S) := e^{-\frac{i}{2}w(\gamma)} \cdot (-1)^{l(S)} \cdot s(z, \gamma), \quad (3.2)$$

where $w(\gamma)$ denotes the winding (i.e., the increment of the argument of the tangent vector) along γ . Then, we define a **spinor observable** on the double cover $\tilde{\Omega}^\delta$ as

$$F_\varpi(a, z) := i\eta_a \cdot \sum_{S \in \text{Conf}_{a,z}(\Omega^\delta)} W_\varpi(z, S) x^{|S|}. \quad (3.3)$$

Remark 3.2. (i) $W_\varpi(z, S)$ does not depend on the way one chooses the decomposition of a given configuration S into loops and the path γ . The proof is elementary, and we leave it to the reader. Note that it is sufficient to check that the second factor $(-1)^{l(S)}s(z, \gamma)$ is independent of a decomposition, as the rest is well known (e.g., see [HS10, Lemma 7]).

- (ii) By definition, $F_\varpi(a, z^*) \equiv -F_\varpi(a, z)$, thus we call $F_\varpi(a, z)$ a *spinor*.
- (iii) If ϖ is the trivial cover, then Definition 3.1 reproduces the original construction due to Smirnov (e.g., see [CS12, eq. (2.10)]). We denote this observable by $F_0(a, z)$ and the corresponding complex phase by $W_0(z, S)$.

The most important “discrete” properties of the observable (3.3) are revealed in Theorem 3.3 below, which states its *s-holomorphicity* (see [Smi10, Section 3] or [CS12, Definition 3.1]) and describes the boundary conditions. We introduce the following notation: given a vertex $v \in \mathcal{V}(\Omega^\delta)$, we consider four nearby corners of faces incident to v , and identify them with the points $v_k := v + e^{i\pi(2k+1)/4} \cdot \delta/2\sqrt{2}$, $k = 0, 1, 2, 3$. We denote sets of all corners of Ω^δ and its double cover $\tilde{\Omega}^\delta$ by $\Upsilon(\Omega^\delta)$ and $\Upsilon(\tilde{\Omega}^\delta)$, respectively. Similarly to (3.1), for a corner $c = v_k \in \Upsilon(\tilde{\Omega}^\delta)$ we set

$$\eta_c := (i(c-v)/|c-v|)^{-\frac{1}{2}} := e^{-i\pi(2k+1)/8}$$

(again, the particular choice of square root signs is unimportant, so we fix it once forever for each of four possible orientations of $c-v$). We denote by

$$\mathbf{Pr}_{\eta_c}(F) := \operatorname{Re}(\overline{\eta_c} F) \eta_c = \frac{1}{2}(F + \eta_c^2 \overline{F})$$

the orthogonal projection of a complex number $F \in \mathbb{C}$ onto the line $\eta_c \mathbb{R}$.

Theorem 3.3. *For any corner $c \in \Upsilon(\tilde{\Omega}^\delta)$ formed by edges or half-edges $z', z'' \in \mathcal{E}(\tilde{\Omega}^\delta)$, one has*

$$\mathbf{Pr}_{\eta_c}(F_\varpi(a, z')) = \mathbf{Pr}_{\eta_c}(F_\varpi(a, z'')). \quad (3.4)$$

Moreover, if $b \in \partial\tilde{\Omega}^\delta \setminus \{a, a^\}$ is a boundary half-edge, then $F_\varpi(a, b) \parallel \eta_b$, i.e.,*

$$\mathbf{Pr}_{i\eta_b}(F_\varpi(a, b)) = 0. \quad (3.5)$$

Remark 3.4. Since our observables satisfy $F_\varpi(a, z^*) \equiv -F_\varpi(a, z)$, the identities (3.4) at two corners c, c^* such that $\varpi(c) = \varpi(c^*)$ are equivalent. The same is fulfilled for the boundary condition (3.5).

Proof. Let v denotes the vertex incident to both z' and z'' . There exists a natural bijection $\Pi : \operatorname{Conf}_{a, z'}(\Omega^\delta) \rightarrow \operatorname{Conf}_{a, z''}(\Omega^\delta)$, provided by taking “xor” of a generalized configuration S with two half-edges $\varpi(vz')$ and $\varpi(vz'')$. The well known proof of the theorem for the trivial cover (e.g., see [CS12, Proposition 2.5] or [HS10, Lemma 45]) assures that, for any $S \in \operatorname{Conf}_{a, z'}(\Omega^\delta)$,

$$\mathbf{Pr}_{\eta_c}(W_0(z', S)x^{|S|}) = \mathbf{Pr}_{\eta_c}(W_0(z'', \Pi(S))x^{|\Pi(S)|}).$$

Clearly, the same holds true with W_0 replaced by W_ϖ , unless Π changes the number of non-trivial loops $l(S)$ or $s(z', \gamma_S) \neq s(z'', \gamma_{\Pi(S)})$. However, it is easy to see that Π always preserves the factor $(-1)^l \cdot s$: for instance, if there was a non-trivial loop in S that disappeared in $\Pi(S)$, then this loop has become a part of the path $\gamma_{\Pi(S)}$, leading to the simultaneous change of the sign s .

To derive the boundary condition (3.5), it is sufficient to note that the winding of any curve γ running from a to b is equal to $(\arg b - (\arg a + \pi))$ modulo 2π . \square

The next proposition relates the boundary values of $F_\varpi(a, \cdot)$ to spin correlations in the Ising model. For a given double cover $\varpi : \tilde{\Omega}^\delta \rightarrow \Omega^\delta$ of a $(k+1)$ -connected domain Ω^δ , let us fix the enumeration of inner boundary components $\gamma_1, \dots, \gamma_k$ so that

ϖ branches around each of $\gamma_1, \dots, \gamma_m$ but not around $\gamma_{m+1}, \dots, \gamma_k$.

For simplicity, below we assume that two marked boundary points a, b belong to the *outer* boundary of $\tilde{\Omega}^\delta$. We denote by Z_{ab} and \mathbb{E}_{ab} the partition function and the expectation in the Ising model with “ $-$ ” boundary conditions on the counterclockwise boundary arc $(\varpi(a)\varpi(b)) \subset \partial\Omega^\delta$, “ $+$ ” on the complementary arc $(\varpi(b)\varpi(a))$, and *monochromatic* on all inner boundary components $\gamma_1, \dots, \gamma_k$. Recall that these boundary conditions require the spin to be constant along each γ_j . We denote this (random) spin by $\sigma(\gamma_j)$.

Remark 3.5. If some hole γ_j is a single face, then we do not impose any boundary condition there and $\sigma(\gamma_j)$ is just a spin assigned to this face.

Proposition 3.6. *If $a, b \in \partial\tilde{\Omega}^\delta$ belong to the outer boundary and $\varpi(a) \neq \varpi(b)$, then*

$$F_\varpi(a, b) = \pm \eta_b \cdot Z_{ab} \mathbb{E}_{ab}[\sigma(\gamma_1)\sigma(\gamma_2)\dots\sigma(\gamma_m)] \quad (3.6)$$

(the choice of sign is explained in Remark 3.7). In particular, $F_0(a, b) = \pm \eta_b \cdot Z_{ab}$. Moreover,

$$F_\varpi(a, a) = i\eta_a \cdot Z_+ \mathbb{E}_+[\sigma(\gamma_1)\sigma(\gamma_2)\dots\sigma(\gamma_m)]. \quad (3.7)$$

Remark 3.7. The sign \pm in (3.6) depends on particular choices of η_a, η_b and the sheets of a, b on $\tilde{\Omega}^\delta$. One way to fix it is as follows: let b be such that the counterclockwise boundary arc $(\varpi(a)\varpi(b)) \subset \partial\Omega^\delta$ lifts to $(ab) \subset \partial\tilde{\Omega}^\delta$ (otherwise, consider $F_\varpi(a, b^*) = -F_\varpi(a, b)$). Then, one can replace $\pm\eta_b$ in (3.6) by $-\eta_a e^{-\frac{i}{2}w_{ab}}$, where w_{ab} denotes the winding of the arc $(\varpi(a)\varpi(b))$.

Proof. The second identity is clear from the definition (3.3), since each configuration $S \in \text{Conf}_{a,a}(\Omega^\delta) = \text{Conf}_+(\Omega^\delta)$ contributes the same amount $i\eta_a(-1)^{l(S)}x^{|S|}$ to both sides of (3.7). To prove (3.6), note that each $S \in \text{Conf}_{a,b}(\Omega^\delta)$ contributes $\pm\eta_b x^{|S|}$ to both sides, thus we only need to check that all the signs are the same. Given a configuration S , decompose it into a path $\gamma : \varpi(a) \rightsquigarrow \varpi(b)$ and a collection of loops. A loop contributes to $l(S)$ (i.e., changes the sheet in $\tilde{\Omega}^\delta$) if and only if it has an odd number of components $\gamma_1, \dots, \gamma_m$ inside. Hence, removing all those loops from the configuration results in the same sign change $(-1)^{l(S)}$ for both sides. Removing the other loops (having an even number of those γ_j ’s inside) does not change the signs. After removing all loops, moving γ across any of γ_j will change $\sigma(\gamma_j)$ and the sheet on which the lifting of γ ends (i.e., the sign $s(b, \gamma)$), again resulting in (-1) factor at both sides. If finally γ goes along the counterclockwise arc $(\varpi(a)\varpi(b))$, then S contributes $x^{|S|}$ to $Z_{ab}\mathbb{E}_{ab}[\sigma(\gamma_1)\sigma(\gamma_2)\dots\sigma(\gamma_m)]$, while its contribution to the left-hand side is equal to $i\eta_a e^{-\frac{i}{2}(w_{ab}-\pi)}x^{|S|} = -\eta_a e^{-\frac{i}{2}w_{ab}}x^{|S|} = \pm\eta_b x^{|S|}$, if γ lifted to the double cover ends at b , and $\eta_a e^{-\frac{i}{2}w_{ab}}x^{|S|}$, if it ends at b^* . \square

3.2. Continuous spinors and convergence results. In this section we introduce continuous counterparts of the discrete holomorphic spinor observables, which we will later prove to be scaling limits thereof. For a moment, let us assume that Ω is a bounded finitely connected domain whose boundary components are single points $\gamma_1 = \{w_1\}, \dots, \gamma_s = \{w_s\}$ and *smooth* curves $\gamma_0, \gamma_{s+1}, \dots, \gamma_k$. Given a double cover $\varpi : \tilde{\Omega} \rightarrow \Omega$ and a point $a \in \partial\tilde{\Omega} \setminus \{w_1, \dots, w_s\}$, we denote by $f_\varpi^\Omega(a, \cdot) : \tilde{\Omega} \rightarrow \mathbb{C}$ (or just f_ϖ for shortness) an analytic function which does not vanish identically and satisfies the following properties:

$$(a^\circ) \quad f_\varpi(a, z) \equiv -f_\varpi(a, z^*) \text{ everywhere in } \tilde{\Omega};$$

- (b°) $f_\varpi(a, \cdot)$ is continuous up to $\partial\tilde{\Omega}$ except, possibly, at the single-point boundary components and at a , and satisfies Riemann boundary conditions

$$f_\varpi(a, z)\sqrt{in_z} \in \mathbb{R}, \quad z \in \partial\tilde{\Omega} \setminus \{a, w_1, \dots, w_s\},$$

where n_z denotes the *outer* normal to Ω at z ;

- (c°) for each single-point boundary component $\{w_j\}$ the following is fulfilled: if ϖ branches around w_j , then there exists a *real* constant c_j such that

$$f_\varpi(a, z) = \frac{\sqrt{i}c_j}{\sqrt{z-w_j}} + O(1) \quad \text{as } z \rightarrow w_j;$$

otherwise f_ϖ is bounded near w_j , and thus has a removable singularity there;

- (d°) in a vicinity of the point a , one has

$$f_\varpi(a, z) = \frac{\sqrt{in_a}c^a}{z-a} + O(1) \quad \text{as } z \rightarrow a$$

for a *real* constant c^a .

The properties (b°)–(d°) should be thought of as natural continuous analogues of those satisfied by $F_\varpi(a, \cdot)$ on the discrete level. Namely, (b°) corresponds to the boundary condition (3.5); (c°) turns out to be the correct formulation of this boundary condition for microscopic holes; and (d°) states that f_ϖ has the simplest possible behaviour near a , which roughly resembles the fact that $F_\varpi(a, \cdot)$ fails to satisfy (3.5) at one boundary edge only.

Remark 3.8. Lemma 3.10 below shows that properties (a°)–(d°) define the function f_ϖ uniquely up to multiplication by a real constant; moreover, $c^a \neq 0$ unless f_ϖ vanishes identically. Sometimes it is convenient to fix this constant so that

$$c^a = 1. \tag{3.8}$$

However, below we also work with non-smooth domains, for which c^a is not well-defined; therefore, we prefer to keep the multiplicative normalization of f_ϖ unfixed.

The boundary value problem (a°)–(d°) is not easy to analyse directly. However, the following trick relates it to a much simpler Dirichlet-type problem: given a spinor f_ϖ , denote

$$h_\varpi(z) := \operatorname{Im} \int^z (f_\varpi(\zeta))^2 d\zeta, \quad z \in \Omega.$$

Note that the function $(f_\varpi(\zeta))^2 = (f_\varpi(\zeta^*))^2$ is analytic in Ω , so h_ϖ is locally well-defined and harmonic.

Lemma 3.9. *A holomorphic spinor f_ϖ satisfies the conditions (b°)–(d°) if and only if h_ϖ is a single-valued harmonic function satisfying the following properties:*

- (b°_h) h_ϖ is continuous up to $\partial\Omega \setminus \{a, w_1, \dots, w_s\}$; moreover, $h_\varpi \equiv \text{const}$ and $\partial_n h_\varpi \leq 0$ on all macroscopic inner boundary components $\gamma_{s+1}, \dots, \gamma_k$ and the outer boundary of Ω , where ∂_n stands for the outer normal derivative;
- (c°_h) h_ϖ is bounded from above near single-point boundary components $\{w_j\}$;
- (d°_h) h_ϖ is bounded from below near the point a .

Proof. Let f_ϖ be a holomorphic spinor such that (b°)–(d°) are fulfilled. On the smooth boundary $\partial\Omega \setminus \{a, w_1, \dots, w_s\}$, one can reformulate the boundary condition (b°) as $(f_\varpi(z))^2 \cdot in_z \geq 0$, which is equivalent to say that $\partial_{in} h_\varpi \equiv 0$ and $\partial_n h \leq 0$, as

stated by (b_h°) . Further, (c°) is equivalent to say that $h(z) = c_j^2 \log |z - w_j| + O(1)$ as $z \rightarrow w_j$, hence (c_h°) holds true. In particular, h_ϖ is single-valued in Ω as it is single-valued near each of w_j and constant along each of macroscopic boundary components. Similarly, (d°) can be rewritten as $h(z) = -(c^a)^2 \operatorname{Re} \frac{n_a}{z-a} + O(1)$ as $z \rightarrow a$, which is equivalent to (d_h°) .

Vice versa, if h_ϖ satisfies Dirichlet boundary conditions on smooth macroscopic boundary components, then $f_\varpi = [\partial_y h_\varpi + i \partial_x h_\varpi]^{1/2}$ is continuous up to $\partial\tilde{\Omega} \setminus \{a, w_1, \dots, w_s\}$. Then, one can easily apply the similar arguments as above to deduce (b°) – (d°) from (b_h°) – (d_h°) . \square

Lemma 3.10. *The holomorphic spinor $f_\varpi^\Omega(a, \cdot)$ with the properties (a°) – (d°) above, if exists, is unique up to multiplication by a positive constant. Moreover, if $\phi : \Omega \rightarrow \Omega'$ is a conformal map, then, again up to multiplicative constants,*

$$f_\varpi^\Omega(a, \cdot) = (\phi'(z))^{\frac{1}{2}} f_{\varpi'}^{\Omega'}(\phi(a), \phi(\cdot)) \quad (3.9)$$

and $h_\varpi^\Omega(a, \cdot) = h_{\varpi'}^{\Omega'}(\phi(a), \phi(\cdot))$, with ϖ' being the pushforward of ϖ by ϕ .

Proof. Suppose that f_1, f_2 both satisfy (a°) – (d°) . Then, one can compose a linear combination $f := c_2^a f_1 - c_1^a f_2$ with non-zero coefficients, such that f satisfies (a°) – (c°) and is *bounded* in a neighborhood of a . As above, define a harmonic function $h(z) := \operatorname{Im} \int^z (f(\zeta))^2 d\zeta$, and note that it is continuous up to $\partial\Omega \setminus \{w_1, \dots, w_s\}$.

As (b°) implies $\partial_n h \leq 0$ everywhere on macroscopic boundary components, the function h cannot attain its maximum there. Also, (c°) says that h is bounded from above in a neighborhood of each w_j . Then, the maximum principle gives $h \equiv \text{const}$ and $f \equiv 0$ everywhere in Ω , i.e., f_1 and f_2 are proportional to each other.

For the second claim, it is sufficient to check the properties (a°) – (d°) for the right-hand side of (3.9) and apply uniqueness, which we leave to the reader. \square

The existence of a non-trivial solution f_ϖ to the above boundary value problem will follow from Theorem 3.13; we also refer the reader to [HP12] for a purely analytic techniques developed for boundary problems of this kind.

The conformal covariance property (3.9) immediately allows one to extend the definition of f_ϖ^Ω to non-smooth or unbounded domains:

Definition 3.11. *If Ω is a finitely connected bounded domain, such that $\partial\Omega$ consists of smooth arcs and single points, we define $f_\varpi^\Omega(a, \cdot)$ as the unique, up to a multiplicative constant, non-zero solution to the boundary value problem (a°) – (d°) . Otherwise we define it by (3.9), where Ω' is any smooth bounded domain.*

Further, we choose a harmonic function $h_\varpi^\Omega(a, z) := \int^z (f_\varpi^\Omega(a, \zeta))^2 d\zeta$ so that $h_\varpi^\Omega(a, \cdot) \equiv 0$ on the boundary component of Ω containing the point a , thus h_ϖ^Ω is defined up to a multiplicative constant as well.

Remark 3.12. An equivalent definition of $h_\varpi^\Omega(a, \cdot)$ for non-smooth domains would be to impose the conditions (b_h°) – (d_h°) given in Lemma 3.9 and the condition that $f_\varpi := [\partial_y h_\varpi + i \partial_x h_\varpi]^{1/2}$ is a spinor on $\tilde{\Omega}$. Indeed, the only condition in Lemma 3.9 that relies on smoothness of $\partial\Omega$ is $\partial_n h_\varpi \leq 0$ which can be reformulated in the following conformally invariant way:

$$\text{there is no } b \in \partial\Omega \setminus \{a, w_1, \dots, w_s\} \text{ such that } h_\varpi(\cdot) < h_\varpi(b) \text{ near } b. \quad (3.10)$$

For our convergence results, we assume that discrete domains Ω^δ approximate Ω in the Carathéodory topology, see [Pom92] or [CS11, Section 3.2]. The reader

unfamiliar with that notion can think of the (stronger) Hausdorff convergence. To simplify notation, we also assume that Ω^δ has the same topology as Ω . The first theorem says that discrete spinors defined in Section 3.1 (with respect to a fixed double cover ϖ of the refining domains Ω^δ) are uniformly close to their continuous counterparts in the bulk of Ω .

Theorem 3.13. *Suppose that Ω^δ is a sequence of discrete domains of mesh size δ approximating (in the sense of Carathéodory) a continuous finitely connected domain Ω , and that $a^\delta \in \partial\tilde{\Omega}^\delta$ tends to some $a \in \partial\tilde{\Omega}$ which is not a single-point boundary component. Then, there exists a sequence of normalizing factors $\beta(\delta) = \beta(\delta; \Omega^\delta, a^\delta, \varpi)$ such that*

$$\beta(\delta)F_{\varpi}(a^\delta, \cdot) \rightarrow f_{\varpi}^\Omega(a, \cdot), \quad \delta \rightarrow 0,$$

uniformly on compact subsets of Ω .

Proof. See Section 4. □

When extending this convergence to the boundary, we will impose additional regularity assumptions:

Definition 3.14. *We say that a sequence of discrete domains Ω^δ with marked boundary points b^δ approximating a planar domain Ω with a marked boundary point b is regular at b , if*

- *near b , the boundary $\partial\Omega$ locally coincide with a horizontal or vertical line;*
- *there exist $s, t > 0$, such that, for any δ , $V(\Omega^\delta)$ contains a discrete rectangle*

$$R^\delta(s, t) := \{\delta \cdot (k + i(l + \frac{1}{2})) : -s \leq k\delta \leq s, 0 \leq l\delta \leq t\},$$

shifted and rotated so that b^δ is the midpoint of its boundary side, and $\partial\Omega^\delta$ coincides with that side in the s -neighborhood of b^δ .

Remark 3.15. In fact, all our results can be directly extended to the case of a straight, but not necessarily vertical or horizontal boundary (cf. [CS12, Theorem 5.6]). Some additional technicalities are required to prove this result in the full generality: note that f_{ϖ}^Ω is not even continuous or bounded on the non-smooth boundary, so one is forced to work with ratios, as, e.g., in Theorem 3.16 below.

Theorem 3.16. *Let ϖ_1, ϖ_2 be two double covers of a bounded finitely connected domain Ω with two marked points a, b on the outer boundary component, and let Ω^δ converge to Ω in the Carathéodory sense, a^δ, b^δ be boundary points converging to a, b as $\delta \rightarrow 0$, and this convergence is regular at a and b . Then,*

$$\frac{F_{\varpi_1}(a^\delta, b^\delta)F_{\varpi_2}(a^\delta, a^\delta)}{F_{\varpi_1}(a^\delta, a^\delta)F_{\varpi_2}(a^\delta, b^\delta)} \rightarrow \frac{f_{\varpi_1}(a, b)}{f_{\varpi_2}(a, b)} \quad (3.11)$$

where both f_{ϖ_1} and f_{ϖ_2} are normalized by (3.8).

Remark 3.17. Formally, above we should have used different notation a_1^δ, a_2^δ etc, to denote points lying on different double covers, with $\varpi_1(a_1^\delta) = \varpi_2(a_2^\delta)$. We prefer to keep a simpler notation for the shortness.

Proof. See Section 4. □

In the next corollary, let ϖ be a fixed double cover of Ω , and $\gamma_1, \dots, \gamma_m$ be those inner components of $\partial\Omega$ for which ϖ branches around γ_j . Denote by γ_j^δ the corresponding components of $\partial\Omega^\delta$.

Corollary 3.18. *Under the conditions of Theorem 3.16, as $\delta \rightarrow 0$, one has*

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(\gamma_1^\delta) \dots \sigma(\gamma_m^\delta)]}{\mathbb{E}_+[\sigma(\gamma_1^\delta) \dots \sigma(\gamma_m^\delta)]} \rightarrow \frac{f_\varpi^\Omega(a, b)}{f_0^\Omega(a, b)} =: \vartheta_{ab}^\Omega(\gamma_1, \dots, \gamma_m), \quad (3.12)$$

where both f_ϖ^Ω and f_0^Ω are normalized by (3.8). The limit $\vartheta_{ab}^\Omega(\gamma_1, \dots, \gamma_m)$ is a conformal invariant of $(\Omega; a, b)$.

Proof. Denote $\sigma(\Gamma) := \sigma(\gamma_1^\delta)\sigma(\gamma_2^\delta) \dots \sigma(\gamma_m^\delta)$. Due to Proposition 3.6, one has

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(\Gamma)]}{\mathbb{E}_+[\sigma(\Gamma)]} = \frac{Z_{a^\delta b^\delta} \mathbb{E}_{a^\delta b^\delta}[\sigma(\Gamma)]}{Z_+ \mathbb{E}_+[\sigma(\Gamma)]} \cdot \frac{Z_+}{Z_{a^\delta b^\delta}} = \frac{F_\varpi(a^\delta, b^\delta)}{F_\varpi(a^\delta, a^\delta)} \cdot \frac{F_0(a^\delta, a^\delta)}{F_0(a^\delta, b^\delta)}.$$

Thus, (3.12) immediately follows from Theorem 3.16. The limit is conformally invariant due to the same conformal covariance property (3.9) of both f_ϖ^Ω and f_0^Ω (observe also that the coefficients c_ϖ^a, c_0^a for both f_ϖ and f_0 change by the same factor $|\phi'(a)|^{-1/2}$ when applying (3.9)). \square

Remark 3.19. (i) Corollary 5.10 gives a generalization of this result for the case of $2n+2$ marked boundary points and “ $+/-/\dots/+/-$ ” boundary conditions.

(ii) Let a third point d be marked on the outer boundary of Ω and the convergence of Ω^δ to Ω is regular at d as well. Then,

$$\frac{Z_{a^\delta b^\delta}}{Z_{a^\delta d^\delta}} = \frac{|F_0(a^\delta, b^\delta)|}{|F_0(a^\delta, d^\delta)|} \rightarrow \frac{|f_0(a, b)|}{|f_0(a, d)|}, \quad (3.13)$$

and this limit is a conformal covariant of the multiply connected domain $(\Omega; a, b, d)$ (namely, it is multiplied by the factor $|\phi'(b)|^{1/2}|\phi'(d)|^{-1/2}$ when applying a conformal map ϕ). For simply connected Ω 's, this is given by [CS12, Corollary 5.7], and we give a proof for multiply connected domains in the end of Section 4.

4. PROOF OF THE MAIN CONVERGENCE THEOREMS

In this section we prove Theorems 3.13 and 3.16 following the scheme developed in [CS12] for simply connected domains. First of all, in order to transform the boundary conditions (3.5) to the Dirichlet ones, we consider a discrete integral $H^\delta := \text{Im} \int (F_\varpi^\delta(z))^2 d^\delta z$. Further, we observe that, under a proper normalization, the functions H^δ and F_ϖ^δ have non-trivial subsequential limits on compact subsets of Ω as $\delta \rightarrow 0$. We then show that any such subsequential limit is a solution to the boundary value problem (a $^\circ$)-(d $^\circ$), and Lemma 3.10 guarantees that all those limits are the same, concluding the proof of Theorem 3.13. Finally, we treat the behaviour of F_ϖ^δ near the boundary points b and a to prove Theorem 3.16.

For technical purposes, we extend our domain slightly: denote by $\partial\mathcal{F}(\Omega^\delta)$ and $\partial\mathcal{V}(\Omega^\delta)$ the subsets of faces and vertices that are adjacent but do not belong to $\mathcal{F}(\Omega^\delta)$ and $\mathcal{V}(\Omega^\delta)$, respectively. More precisely, $\partial\mathcal{V}(\Omega^\delta)$ can be identified with the set $\mathcal{E}_{\text{bd}}(\Omega^\delta)$ of boundary half-edges and should be formally considered as a set of pairs $\{(v; e) : e = (v_{\text{int}}v), v \notin \mathcal{V}(\Omega^\delta), v_{\text{int}} \in \mathcal{V}(\Omega^\delta)\}$ (e.g., see [CS11, Section 2.1]), and $\partial\mathcal{F}(\Omega^\delta)$ should be treated in the same way. We set

$$\bar{\mathcal{V}}(\Omega^\delta) := \mathcal{V}(\Omega^\delta) \cup \partial\mathcal{V}(\Omega^\delta) \quad \text{and} \quad \bar{\mathcal{F}}(\Omega^\delta) := \mathcal{F}(\Omega^\delta) \cup \partial\mathcal{F}(\Omega^\delta).$$

We work with the discrete spinor $F(z) = F_\varpi(a; z)$ defined on a double cover $\tilde{\Omega}^\delta$ of a discrete multiply connected domain Ω^δ , which we do not include in the notation unless needed. Recall that it is s-holomorphic (i.e., satisfies (3.4)) and obeys the boundary conditions (3.5) at all boundary half-edges e , except for one

edge a on the boundary. We denote the corresponding vertex of $\partial\mathcal{V}(\Omega^\delta)$ by v_a . Observe also that F_ϖ is not identically zero, since the positivity of spin correlations and Proposition 3.6 yield $(i\eta_a)^{-1}F_\varpi(a; a) > 0$. These are the only properties of $F_\varpi(a; \cdot)$ that we will use in this section.

Recall that, in the continuous case, it was proved to be useful to transform the boundary value problem (a $^\circ$)–(d $^\circ$) into a Dirichlet-type one by integrating f_ϖ^2 . The extension of this construction to the discrete setting is delicate, because the square of discrete analytic function need not be discrete analytic. Fortunately, the tools to treat this issue have already been developed in [Smi10, CS12]. Proposition 4.1 below summarizes these tools. Namely, properties (1)–(3) thereof show that one can define the discrete analog of $\text{Im} \int f_\varpi^2$ as a pair of functions H_\bullet and H_\circ , one of which is subharmonic and another superharmonic; properties (4)–(6) handle the boundary conditions, and properties (7),(8) show that these two functions cannot be too far from each other. This allows one to work with a pair H_\bullet, H_\circ as if it was a single harmonic function. Essentially, our analysis is based on a priori bounds for H derived from simple harmonic measure estimates combined with the uniqueness of solution to the boundary value problem (a $^\circ$)–(d $^\circ$).

We define two functions H_\bullet and H_\circ on $\overline{\mathcal{V}}(\Omega^\delta)$ and $\overline{\mathcal{F}}(\Omega^\delta)$, respectively, by the following rule: if $c \in \Upsilon(\Omega^\delta)$ and $e \in \mathcal{E}(\Omega^\delta)$, $v \in \overline{\mathcal{V}}(\Omega^\delta)$, $f \in \overline{\mathcal{F}}(\Omega^\delta)$ are all incident to c , then

$$H_\bullet(v) - H_\circ(f) := \sqrt{2}\delta \cdot |\mathbf{Pr}_{\eta_c}(F(\varpi^{-1}(e)))|^2. \quad (4.1)$$

Thanks to the square, this definition does not depend on the choice of the sheet, and thanks to the basic definition (3.4) of s-holomorphicity, it does not depend on the choice of e between the two edges (or boundary half-edges) adjacent to c .

Proposition 4.1. *The functions H_\bullet and H_\circ obey the following properties:*

- (1) *they are well-defined up to an additive constant;*
- (2) *if $e \in \mathcal{E}(\Omega^\delta)$ is incident to $f, f' \in \overline{\mathcal{F}}(\Omega^\delta)$ and $v, v' \in \overline{\mathcal{V}}(\Omega^\delta)$, then:*

$$H_\bullet(v) - H_\bullet(v') = \text{Im}[(F(\varpi^{-1}(e))^2(v - v'))];$$

$$H_\circ(f) - H_\circ(f') = \text{Im}[(F(\varpi^{-1}(e))^2(f - f'))];$$

- (3) *$\Delta H_\circ(v) \leq 0$ and $\Delta H_\bullet(f) \geq 0$ for all $v \in \mathcal{V}(\Omega^\delta)$, $f \in \mathcal{F}(\Omega^\delta)$, where Δ stands for the discrete Laplacian*

$$(\Delta H)(x) := \frac{1}{4\delta^2} \sum_{x_k \sim x} (H(x_k) - H(x));$$

- (4) *if $v \in \partial\mathcal{V}(\Omega^\delta) \setminus \{v_a\}$, then $H_\bullet(v_{\text{int}}) - H_\bullet(v) \geq 0$;*
- (5) *$H_\circ \equiv \text{const} =: C_j$ along each component of $\partial\mathcal{F}(\Omega^\delta)$ (then we fix an additive constant in the definition of H so that $H_\circ \equiv 0$ on the boundary component that contains a);*
- (6) *one can modify the discrete Laplacian at all vertices $v_{\text{int}} \in \mathcal{V}(\Omega^\delta)$ incident to $v \in \partial\mathcal{V}(\Omega^\delta) \setminus \{v_a\}$ and set values of H_\bullet on $\partial\mathcal{V}(\Omega^\delta) \setminus \{v_a\}$ to be equal to the corresponding C_j , so that (3) and (4) will still hold (moreover, this modification do not destroy any estimates or convergence results for discrete harmonic functions defined on $\mathcal{V}(\Omega^\delta)$);*
- (7) *if $v_1, v_2, v_3, v_4 \in \mathcal{V}(\Omega^\delta)$ and $f_1, f_2, f_3, f_4 \in \overline{\mathcal{F}}(\Omega^\delta)$ are adjacent to some inner face $f \in \mathcal{F}(\Omega^\delta)$ and $m := \min H_\circ(f_j)$, then, for all $j = 1, 2, 3, 4$,*

$$H_\bullet(v_j) - H_\circ(f) \leq \text{const} \cdot (H_\circ(f) - m)$$

with some universal constant;

- (8) if $f_1, f_2, f_3, f_4 \in \overline{\mathcal{F}}(\Omega^\delta)$ and $v_1, v_2, v_3, v_4 \in \overline{\mathcal{V}}(\Omega^\delta)$ are adjacent to some inner vertex $v \in \mathcal{V}(\Omega^\delta)$ and $M = \max H_\bullet(v_j)$, then, for all $j = 1, 2, 3, 4$,

$$H_\bullet(v) - H_\circ(f_j) \leq \text{const} \cdot (M - H_\bullet(v))$$

with some universal constant.

Proof. All these properties are known in the simply connected case (e.g., see [CS12, Section 3]). Since (2), (3), (7) and (8) are local consequences of s-holomorphicity (3.4), they extend immediately to the multiply connected setup. Properties (4) and (5) follow from the boundary condition (3.5) using (2). The property (6) is also a local property of s-holomorphic function satisfying the boundary condition (3.5), see [CS12, Section 3.6] or [DHN11, Proof of Proposition 8]. So we only need to check (1), i.e. that the summation of (4.1) along any loop gives zero. For homotopically trivial loops this follows from the local consistency of the definition (4.1) exactly as in the simply connected case, and extends to loops running around holes by (5). \square

Remark 4.2. Note that we have an immediate corollary of properties (1)–(4): if F is an s-holomorphic spinor satisfying the boundary condition (3.5) *everywhere* on $\partial\Omega^\delta$ including the point a , then $F \equiv 0$. Indeed, (3) implies that the corresponding function H_\bullet attains its maximal value on the boundary, and then $H_\bullet \equiv \text{const}$ due to the property (4) which now holds true everywhere on $\partial\Omega^\delta$. Moreover, similar arguments show that a solution to the discrete boundary value problem (3.4), (3.5) is unique up to a multiplicative constant (the proof mimics [CS12, Remark 5.1]).

In what follows, we denote by $\text{hm}_{\Omega^\delta}^\delta(v, \gamma)$ the *discrete harmonic measure* of a set $\gamma \subset \overline{\mathcal{V}}(\Omega^\delta)$ in the discrete domain Ω^δ viewed from a vertex v . Recall that $\text{hm}_{\Omega^\delta}^\delta(v, \gamma)$ is given by the probability of the event that the simple random walk starting at v hits γ before $\partial\mathcal{V}(\Omega^\delta) \setminus \gamma$. We use the same notation $\text{hm}_{\Omega^\delta}^\delta(f, \gamma)$ for the discrete harmonic measure of a set $\gamma \subset \overline{\mathcal{F}}(\Omega^\delta)$ viewed from a face f . Essentially, we will use only the following elementary properties of hm^δ , which also are fulfilled for the discrete Laplacian modified near the boundary as it is mentioned in Proposition 4.1, property (6) (see [CS12, Section 3.6] or [DHN11, Proof of Proposition 8]):

- *weak Beurling-type estimate*: there exist absolute (i.e., independent of δ , Ω^δ and γ) constants $C, p > 0$ such that the following estimate holds true:

$$\text{hm}_{\Omega^\delta}^\delta(z, \partial\Omega^\delta \setminus \gamma) \leq C \cdot \left[\frac{\text{dist}(z; \gamma)}{\text{dist}_{\Omega^\delta}(z; \partial\Omega^\delta \setminus \gamma)} \right]^p, \quad (4.2)$$

where $\text{dist}_{\Omega^\delta}(z, K)$ means the smallest $r > 0$ such that z and K are connected inside $\Omega^\delta \cap B_r(z)$.

- *uniform estimates for the exit probabilities in rectangles*: if $R^\delta = R^\delta(2s, t)$ is a discretization of the rectangle $(-2s, 2s) \times (0, t)$ with $s \geq t$, γ_0, γ_1 denote correspondingly the bottom and the top side of R^δ , then

$$C_1 \cdot \frac{\text{Im}z}{t} \leq \text{hm}_{R^\delta}^\delta(z, \gamma_1) \leq \text{hm}_{R^\delta}^\delta(z, \partial R^\delta \setminus \gamma_0) \leq C_2 \cdot \frac{\text{Im}z}{t} \quad (4.3)$$

for all $z \in R^\delta(s, t)$, where $C_1, C_2 > 0$ are some absolute constants.

Note that (4.2) and (4.3) hold true for arbitrary isoradial graphs (see [CS11]).

Now we proceed to the definition of normalizing factors $\beta(\delta)$ used in Theorem 3.13. Let a sequence of discrete domains Ω^δ approximate a finitely connected

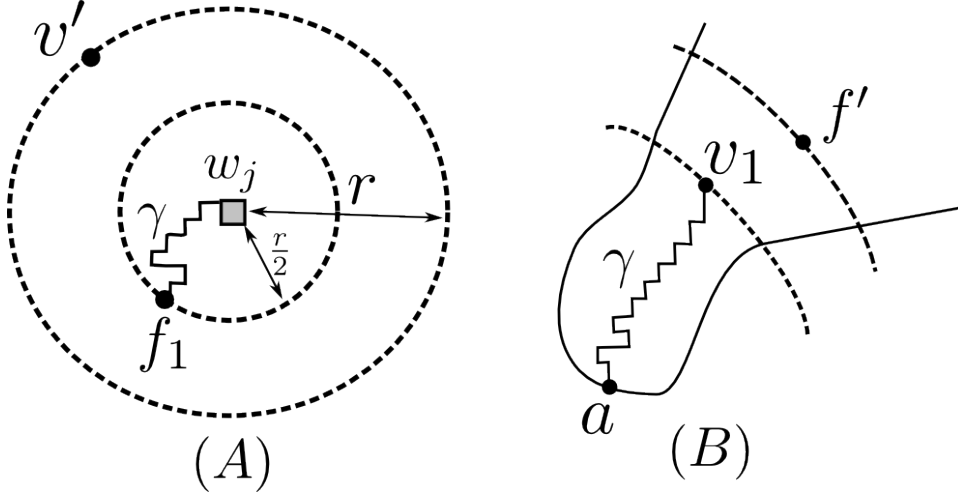


FIGURE 2. (A) The proof of the estimate $\tilde{\beta}_r^\delta \leq \text{const} \cdot \beta_{2r}^\delta$ in Lemma 4.4. If $-H_o$ is big at f , it is also big at vertices near some path $\gamma : f \rightsquigarrow w_j$. Since the harmonic measure of γ seen from v is uniformly bounded from below, $-H_\bullet(v')$ has to be big as well. (B) Similar considerations near a give the estimate $\beta_r^\delta \leq \text{const} \cdot \tilde{\beta}_r^\delta$.

planar domain Ω , whose boundary $\partial\Omega$ consists of single-point inner components $\gamma_1 = \{w_1\}, \dots, \gamma_s = \{w_s\}$ and continua $\gamma_{s+1}, \dots, \gamma_k, \gamma_{k+1}$, where γ_{k+1} denotes the outer boundary of Ω . Let $r_* > 0$ be chosen sufficiently small so that, for any $r \leq r_*$,

$$\Omega(r) := \Omega \setminus [\cup_{j=1}^s B_r(w_j) \cup B_{r,\Omega}(a)]$$

has the same topological structure as Ω , where $B_{r,\Omega}(a)$ denotes the proper connected component of $B_r(a) \cap \Omega$. Further, let $\delta_* = \delta_*(r) > 0$ be chosen small enough so that, for any $\delta \leq \delta_*$, one has $|a^\delta - a| \leq \frac{1}{2}r$ and $\gamma_j^\delta \subset B_{\frac{1}{2}r}(w_j)$ for all $j = 1, \dots, s$.

Let $F^\delta = F_\varpi^\delta(a^\delta, \cdot)$ be the spinor observable (3.3) in Ω^δ and $H_o^\delta, H_\bullet^\delta$ be the corresponding discrete integrals $\text{Im} \int^\delta (F^\delta(z))^2 dz$ defined by (4.1) and Proposition 4.1. We introduce normalizing factors $\beta_r^\delta > 0$ by

$$(\beta_r^\delta)^{-1} := \max_{\Omega^\delta(r)} |H^\delta|, \quad (4.4)$$

where, similarly to the continuous setup, $\Omega^\delta(r) := \Omega^\delta \setminus [\cup_{j=1}^s B_r^\delta(w_j) \cup B_{r,\Omega}^\delta(a)]$ and $B_r^\delta(w_j), B_{r,\Omega}^\delta(a)$ stand for discrete r -neighborhoods of w_j and a in Ω^δ .

Remark 4.3. (i) Note that the right-hand side of (4.4) does not vanish. Indeed, if it did, $F^\delta(a^\delta, \cdot)$ would also vanish identically.

(ii) Since the limiting function f_ϖ has singularities at a and w_j , it is natural to cut them off when taking a maximum to get a correct normalizing factor.

This choice of β_r^δ guarantees that $|\beta_r^\delta H^\delta| \leq 1$ in $\Omega^\delta(r)$. Our next goal is to show that, for a fixed r , the functions $\beta_r^\delta H^\delta$ are uniformly bounded on *any* compact subset of Ω . This is achieved by the following lemma:

Lemma 4.4. *For any $r \leq \frac{1}{2}r_*$, the ratio $\beta_{2r}^\delta/\beta_r^\delta$ is bounded uniformly in $\delta \leq \delta_*(r)$.*

Proof. Denote $\tilde{\Omega}^\delta(r) := \Omega^\delta(r) \setminus B_{2r,\Omega}^\delta(a)$ and

$$(\tilde{\beta}_r^\delta)^{-1} := \max_{\tilde{\Omega}^\delta(r)} |H^\delta|.$$

By definition, $\Omega^\delta(2r) \subset \tilde{\Omega}^\delta(r) \subset \Omega^\delta(r)$, hence $\beta_r^\delta \leq \tilde{\beta}_r^\delta \leq \beta_{2r}^\delta$. We first show that $\beta_{2r}^\delta/\tilde{\beta}_r^\delta$ is uniformly bounded. Recall that H_\bullet^δ is subharmonic, H_\circ^δ is superharmonic, $H_\bullet^\delta(v) \geq H_\circ^\delta(f)$ for adjacent v and f , and $H^\delta \equiv 0$ on the boundary component containing a . Therefore,

$$(\tilde{\beta}_r^\delta)^{-1} = \max \left\{ \max_{\partial\tilde{\Omega}^\delta(r)} H_\bullet^\delta; -\min_{\partial\tilde{\Omega}^\delta(r)} H_\circ^\delta \right\}. \quad (4.5)$$

Note that in fact

$$\max_{\partial\tilde{\Omega}^\delta(r)} H_\bullet^\delta \leq \max_{\partial\Omega^\delta(2r)} H_\bullet^\delta. \quad (4.6)$$

Indeed, the function H_\bullet^δ is subharmonic in $B_{2r}^\delta(w_j) \cap \Omega^\delta$, $j = 1, \dots, s$, and it cannot attain its maximum on the boundary component $\gamma_j^\delta \subset B_{2r}^\delta(w_j)$ because of property (4) of Proposition 4.1 (and, if γ_j^δ consists of one face only, then H_\bullet^δ is subharmonic *everywhere* in $B_{2r}^\delta(w_j)$).

Thus, either $(\tilde{\beta}_r^\delta)^{-1} \leq (\beta_{2r}^\delta)^{-1}$ and then there is nothing to prove, or

$$(\tilde{\beta}_r^\delta)^{-1} = -H_\circ^\delta(f) \quad \text{for some } f \in \partial B_r^\delta(w_j).$$

Since H_\circ^δ is superharmonic, in this case there exists a path of consecutive neighbors $\gamma = \{f = f_1 \sim f_2 \sim \dots\}$ such that $-(\tilde{\beta}_r^\delta)^{-1} = H_\circ^\delta(f_1) \geq H_\circ^\delta(f_2) \geq \dots$. This path can only end up at γ_j^δ , where the superharmonicity of H_\circ^δ fails (see Fig. 2A). Denote by γ' the set of *vertices* adjacent to γ . Then, the property (8) in Proposition 4.1 ensures that

$$H_\bullet^\delta + (\tilde{\beta}_r^\delta)^{-1} \leq \text{const} \cdot ((\beta_{2r}^\delta)^{-1} - H_\bullet^\delta) \quad \text{everywhere on } \gamma',$$

where we have used the estimate $\max_{B_{2r}^\delta(w_j)} H_\bullet^\delta \leq (\beta_{2r}^\delta)^{-1}$ which was explained after (4.6). Hence, for some absolute constant $p_1 > 0$,

$$H_\bullet^\delta \leq -p_1(\tilde{\beta}_r^\delta)^{-1} + (\beta_{2r}^\delta)^{-1} \quad \text{everywhere on } \gamma'.$$

Further, there exists a constant $p_2 > 0$ independent of r and δ such that, for any vertex $v \in \partial B_{2r}^\delta(w_j)$, one has $\text{hm}(v, \gamma') := \text{hm}_{\Omega^\delta(2r) \cup B_{2r}^\delta(w_j)}^\delta(v, \gamma') \geq p_2$ (see Fig. 2A). By subharmonicity of H_\bullet^δ , this implies

$$\begin{aligned} H_\bullet^\delta(v) &\leq (-p_1(\tilde{\beta}_r^\delta)^{-1} + (\beta_{2r}^\delta)^{-1}) \cdot \text{hm}(v, \gamma') + (\beta_{2r}^\delta)^{-1} \cdot (1 - \text{hm}(v, \gamma')) \\ &\leq -p_1 p_2 (\tilde{\beta}_r^\delta)^{-1} + (\beta_{2r}^\delta)^{-1}. \end{aligned}$$

Since $H_\bullet^\delta(v) \geq -(\beta_{2r}^\delta)^{-1}$, we infer that $\beta_{2r}^\delta/\tilde{\beta}_r^\delta \leq 2(p_1 p_2)^{-1}$.

It remains to prove that $\tilde{\beta}_r^\delta/\beta_r^\delta$ is uniformly bounded, which is done by exactly the same argument with the roles of H_\bullet^δ and H_\circ^δ interchanged, inequalities reversed, and a playing the role of w_j . Note that the uniform lower bound for the harmonic measure $\text{hm}(f, \gamma')$ still holds (this time, γ' will be the set of *faces* adjacent to a *vertex* path γ which terminates at a , see Fig. 2B), provided that f is chosen, say, to be the closest face to some fixed $z \in \Omega \setminus B_{2r,\Omega}(a)$ (e.g., see [CS11, Lemma 3.14]). \square

Now we are able to claim precompactness of the families $\{H^\delta\}, \{F_\varpi^\delta\}$ as $\delta \rightarrow 0$.

Lemma 4.5. *Fix some sufficiently small $r > 0$. Then, there exists a subsequence $\delta = \delta_k \rightarrow 0$ such that the functions $\beta_r^\delta H^\delta$ converge to a harmonic function $H : \Omega \rightarrow \mathbb{R}$ uniformly on compact subsets of Ω . Moreover, the functions $\beta_r^\delta (F^\delta)^2$ converge to $F^2 := \partial_y H + i \partial_x H$ uniformly on compact subsets of Ω .*

Proof. Definition of β_r^δ and Lemma 4.4 guarantee that $\beta_r^\delta H^\delta$ are uniformly bounded in $\Omega^\delta(2^{-k}r)$ for any $k \geq 0$, and hence on all compact subsets of Ω . Due to [CS12, Theorem 3.12], the functions $\beta_r^\delta H^\delta$ and the spinors $\sqrt{\beta_r^\delta} F^\delta$ are thus equicontinuous on compact subsets of Ω^δ , and the Arzela-Ascoli theorem implies their subsequential convergence to a continuous function H and a spinor F , respectively. Morera's theorem, together with the discrete holomorphicity of F^δ , implies that F is holomorphic. Since increments of $\beta_r^\delta H^\delta$ are given by discrete integrals $\text{Im} \int^\delta [\beta_r^\delta (F^\delta(z))^2 d^\delta z]$, one has $H = \text{Im} \int (F(z))^2 dz$, so H is a harmonic function defined in Ω . \square

The next step is to show that all these subsequential limits solve the correct boundary value problem. It is convenient to work with the function H . Recall that, due to Proposition 4.1 (properties (5) and (6)), $H^\delta \equiv C_j^\delta$ on each of macroscopic boundary components $\gamma_{s+1}^\delta, \dots, \gamma_{k+1}^\delta$. By definition (4.4) of the normalizing factors β_r^δ , we have $|\beta_r^\delta C_j^\delta| \leq 1$. Thus, taking a subsequence once more, we can assume that

$$\beta_r^\delta C_j^\delta \rightarrow c_j \text{ as } \delta \rightarrow 0, \quad j = s+1, \dots, k+1, \quad (4.7)$$

for some constants c_j such that $|c_j| \leq 1$.

Lemma 4.6. *For any subsequential limit H from Lemma 4.5 satisfying (4.7), the conditions $(b_h^\circ) - (d_h^\circ)$ from Lemma 3.9 hold true. Moreover, $\sup_{\Omega(r)} |H| = 1$.*

Remark 4.7. Below we use the equivalent reformulation (3.10) of the boundary condition $\partial_n H \leq 0$ which does not rely on the smoothness of $\partial\Omega$.

Proof. Property (b_h°) . Our first goal is to prove that H satisfies the Dirichlet boundary conditions on all macroscopic boundary components γ_j , $j = s+1, \dots, k+1$. Fix some small $r' > 0$, and recall that $|\beta_r^\delta H^\delta| \leq C(r')$ everywhere in $\Omega^\delta(r')$, where $C(r')$ does not depend on δ . Due to superharmonicity of H_\circ^δ , for any face $f \in \Omega^\delta(r')$, one has

$$\beta_r^\delta H_\circ^\delta(f) \geq \beta_r^\delta C_j^\delta \cdot \text{hm}_{\Omega^\delta(r')}(f, \gamma_j^\delta) - C(r')(1 - \text{hm}_{\Omega^\delta(r')}(f, \gamma_j^\delta)). \quad (4.8)$$

Similarly, subharmonicity of H_\bullet^δ implies that, for any vertex $v \in \Omega^\delta(r')$, one has

$$\beta_r^\delta H_\bullet^\delta(v) \leq \beta_r^\delta C_j^\delta \cdot \text{hm}_{\Omega^\delta(r')}(v, \gamma_j^\delta) + C(r')(1 - \text{hm}_{\Omega^\delta(r')}(v, \gamma_j^\delta)). \quad (4.9)$$

Now let f and v approximate some point $z \in \Omega(r')$ as $\delta \rightarrow 0$. Since both discrete harmonic measures converge to the continuous one, and $H_\circ^\delta(f) \leq H_\bullet^\delta(v)$ for incident f and v , (4.8) and (4.9) yield

$$|H(z) - c_j \text{hm}_{\Omega(r')}(z, \gamma_j)| \leq C(r')(1 - \text{hm}_{\Omega(r')}(z, \gamma_j)).$$

In particular, $H(z) \rightarrow c_j$ as z tends to γ_j . The boundary condition (4) from Proposition 4.1 survives in the limit and gives (3.10) due to [CS12, Remark 6.3].

Properties (c_h°) and (d_h°) . By subharmonicity of H_\bullet^δ , the inequality $\beta_r^\delta H_\bullet^\delta \leq 1$ extends from $\partial B_r^\delta(w_j) \subset \partial\Omega^\delta(r)$ to the whole discrete disc $B_r^\delta(w_j)$, thus $H \leq 1$ in a vicinity of w_j . Similarly, superharmonicity of H_\circ^δ and the condition $H_\circ^\delta \equiv 0$ on the boundary component containing a imply $H \geq -1$ near a .

Normalization $\sup_{\Omega(r)} |H| = 1$. Recall that, similarly to (4.5), $|\beta_r^\delta H^\delta|$ attains its maximum $\max_{\Omega^\delta(r)} |\beta_r^\delta H^\delta| = 1$ on the boundary

$$\partial\Omega^\delta(r) \subset \partial B_{r,\Omega}^\delta(a) \cup \partial B_r^\delta(w_1) \cup \cdots \cup \partial B_r^\delta(w_s) \cup \gamma_{s+1}^\delta \cup \cdots \cup \gamma_{k+1}^\delta.$$

Note that (4.7) and the property (b_h°) yield

$$\max_{\gamma_{s+1}^\delta \cup \cdots \cup \gamma_{k+1}^\delta} |\beta_r^\delta H^\delta| \rightarrow \max_{\gamma_{s+1} \cup \cdots \cup \gamma_{k+1}} |H|.$$

Moreover, as $\beta_r^\delta H^\delta$ uniformly converge to H on all compact subsets of Ω , one has

$$\max_{\partial B_r^\delta(w_1) \cup \cdots \cup \partial B_r^\delta(w_s)} |\beta_r^\delta H^\delta| \rightarrow \max_{\partial B_r(w_1) \cup \cdots \cup \partial B_r(w_s)} |H|.$$

Finally, the convergence of $\beta_r^\delta H^\delta$ to H inside of Ω also imply

$$\max_{\partial B_{r,\Omega}^\delta(a)} |\beta_r^\delta H^\delta| \rightarrow \max_{\partial B_{r,\Omega}(a)} |H|,$$

since the estimates (4.8),(4.9) guarantee that $|\beta_r^\delta H^\delta|$ are uniformly small near the boundary component of Ω containing a . \square

Proof of Theorem 3.13. We set $\beta(\delta) := \sqrt{\beta_r^\delta}$ for a small *fixed* $r > 0$. By Lemma 4.5, the functions $(\beta(\delta))^2 H^\delta$ and $\beta(\delta) F^\delta$ have subsequential limits H and F , respectively. Lemmas 4.6, 3.9 and 3.10 guarantee that all these possible limits H are the same. Moreover, this unique limit is nontrivial due to the normalization condition in Lemma 4.6, and thus coincides with h_ϖ^Ω normalized so that $\sup_{\Omega(r)} |h_\varpi^\Omega| = 1$. Since $H(z) = \text{Im} \int^z (F(\zeta))^2 d\zeta$, we conclude that $F = f_\varpi^\Omega$. \square

The next lemma shows that the convergence of $\beta(\delta) F^\delta$ holds true at the boundary point b (in fact, it follows from our proof that this convergence is uniform up to straight parts of the boundary $\partial\Omega$). Note that a similar result has been obtained in [CS12, Theorem 5.6] under milder assumptions. For convenience of the reader, we give a shorter proof here using our (stronger) regularity assumptions for Ω^δ near b .

Lemma 4.8. *Suppose that, under the conditions of Theorem 3.13, we are given a sequence of marked points $b^\delta \in \partial\Omega^\delta$, $b^\delta \rightarrow b$, and the convergence $\Omega^\delta \rightarrow \Omega$ is regular at b . Then, $\beta(\delta) |F^\delta(b^\delta)| \rightarrow |f_\varpi^\Omega(b)|$ as $\delta \rightarrow 0$.*

Proof. Below we assume that Ω and Ω^δ are shifted and rotated so that $b^\delta = b = 0$, Ω^δ contains a discrete rectangle $R^\delta(s, t)$ for some fixed $s, t > 0$, and $\partial\Omega^\delta$ locally coincides with the boundary $\partial\mathbb{C}_+^\delta$ of the discrete upper half-plane \mathbb{C}_+^δ (see Definition 3.14). We also assume that the additive normalization of the functions H^δ is chosen so that they vanish on the macroscopic boundary component containing b .

Recall that, on compact subsets of Ω , $\beta(\delta) F^\delta$ and $(\beta(\delta))^2 H^\delta$ converge to the properly normalized functions f_ϖ and h_ϖ , respectively. Moreover, the uniform estimates (4.8),(4.9) guarantee that the convergence $(\beta(\delta))^2 H^\delta \rightarrow h_\varpi$ remains true up to $\partial\Omega \setminus \{a, w_1, \dots, w_s\}$. In particular, the functions $(\beta(\delta))^2 H^\delta$ converge to h_ϖ uniformly in the fixed rectangle $R(s, t)$ around b .

Let

$$\mu := (f_\varpi(0))^2 = \partial_y h_\varpi(0)$$

and

$$\begin{cases} \mathcal{H}_\circ^\delta(f) := \text{Im} f, & \text{for faces } f \in \mathbb{C}_+^\delta, \\ \mathcal{H}_\bullet^\delta(v) := \text{Im } v + \frac{\delta}{\sqrt{2}}, & \text{for vertices } v \in \mathbb{C}_+^\delta \end{cases}$$

(note that \mathcal{H}^δ can be defined by (4.1) starting with the constant s-holomorphic function $\mathcal{F}^\delta \equiv 1$ which satisfies the boundary conditions (3.5) on $\partial\mathbb{C}_+^\delta$). Then, for any $\varepsilon > 0$, one can find a small $d > 0$ such that

$$|(\beta(\delta))^2 H^\delta - \mu \mathcal{H}^\delta| \leq \varepsilon d \text{ everywhere in } R^\delta := R^\delta(2d, d)$$

for all sufficiently small δ 's. Since both \mathcal{H}_\circ^δ and $\mathcal{H}_\bullet^\delta$ are discrete harmonic and satisfy the same Dirichlet boundary conditions as H^δ near b^δ , the sub- and super-harmonicity of H_\bullet^δ and H_\circ^δ implies that

$$\begin{aligned} (\beta(\delta))^2 H_\circ^\delta - \mu \mathcal{H}_\circ^\delta &\geq -\varepsilon d \cdot \text{hm}_{R^\delta}(\cdot, \partial R^\delta \setminus \partial\mathbb{C}_+^\delta), \\ (\beta(\delta))^2 H_\bullet^\delta - \mu \mathcal{H}_\bullet^\delta &\leq \varepsilon d \cdot \text{hm}_{R^\delta}(\cdot, \partial R^\delta \setminus \partial\mathbb{C}_+^\delta) \end{aligned} \quad \text{everywhere in } R^\delta. \quad (4.10)$$

Let $v_b := \frac{1}{2}i\delta$ be the inner vertex of the boundary half-edge b (see notation on Fig. 3A). Estimating the discrete harmonic measure $\text{hm}_{R^\delta}(v_b, \partial R^\delta \setminus \partial\mathbb{C}_+^\delta)$ by (4.3), one obtains

$$\sqrt{2}\delta \cdot |\beta(\delta)F^\delta(b) \cos \frac{\pi}{8}|^2 = H_\bullet^\delta(v_b) \leq \mu \mathcal{H}_\bullet^\delta(v_b) + \text{const} \cdot \varepsilon \delta.$$

Since $\mathcal{H}_\bullet^\delta(v_b) = \frac{\delta}{2} + \frac{\delta}{\sqrt{2}} = (\sqrt{2} \cos^2 \frac{\pi}{8})\delta$, this gives

$$|\beta(\delta)F^\delta(b)|^2 \leq \mu + \text{const} \cdot \varepsilon.$$

Note that this bound also holds true for the value $|\beta(\delta)F^\delta(b')|^2$, where $b' := \delta$ denotes the neighboring boundary edge and $v'_b := (1 + \frac{1}{2}i)\delta$ is the corresponding inner vertex (see notation on Fig. 3A). Now let $b'' := (\frac{1}{2} + \frac{1}{2}i)\delta$ be the midpoint of the edge $(v_b v'_b)$, and $f_b := (\frac{1}{2} + i)\delta$ be an inner face incident to both v_b and v'_b . Using (4.10) and estimating $\text{hm}_{R^\delta}(f_b, \partial R^\delta \setminus \partial\mathbb{C}_+^\delta)$ by (4.3), one obtains

$$\text{Re}(\beta(\delta)F^\delta(b''))^2 \cdot \delta = H_\circ^\delta(f_b) \geq \mu\delta - \text{const} \cdot \varepsilon\delta.$$

Note that for any complex number $\xi \in \mathbb{C}$ one has

$$\text{Re}(\xi^2) \leq [2 \cos^2 \frac{\pi}{8}]^{-1} \cdot (|\text{Pr}_{e^{-i\pi/8}}(\xi)|^2 + |\text{Pr}_{e^{i\pi/8}}(w)|^2).$$

Therefore, we arrive at the inequalities

$$\begin{aligned} \mu - \text{const} \cdot \varepsilon &\leq \text{Re}(\beta(\delta)F^\delta(b''))^2 \\ &\leq [2 \cos^2 \frac{\pi}{8}]^{-1} \cdot (|\text{Pr}_{e^{-i\pi/8}}(\beta(\delta)F^\delta(b''))|^2 + |\text{Pr}_{e^{i\pi/8}}(\beta(\delta)F^\delta(b''))|^2) \\ &= [2 \cos^2 \frac{\pi}{8}]^{-1} \cdot (|\text{Pr}_{e^{-i\pi/8}}(\beta(\delta)F^\delta(b))|^2 + |\text{Pr}_{e^{i\pi/8}}(\beta(\delta)F^\delta(b'))|^2) \\ &= \frac{1}{2}(|\beta(\delta)F^\delta(b)|^2 + |\beta(\delta)F^\delta(b')|^2) \leq \mu + \text{const} \cdot \varepsilon. \end{aligned}$$

Since ε can be chosen arbitrary small, this yields $\lim_{\delta \rightarrow 0} |\beta(\delta)F^\delta(b)|^2 = \mu$. \square

Finally, we work out the relation of the values of discrete observables F^δ at the points a^δ to the growth rate of their limit f_ϖ near a .

Lemma 4.9. *Under the conditions of Theorem 3.13, assume also that the convergence $\Omega^\delta \rightarrow \Omega$ is regular at a . For $j = 1, 2$, let ϖ_j be double covers of Ω^δ , $\beta_j(\delta)F_{\varpi_j}^\delta \rightarrow f_{\varpi_j}$, and c_j^a denote the corresponding coefficients in the expansions of f_{ϖ_j} near a (see (d°) in the definition of f_{ϖ_j} given in Sect. 3.2). Then,*

$$\frac{\beta_1(\delta)F_{\varpi_1}^\delta(a^\delta)}{\beta_2(\delta)F_{\varpi_2}^\delta(a^\delta)} \rightarrow \frac{c_1^a}{c_2^a}. \quad (4.11)$$

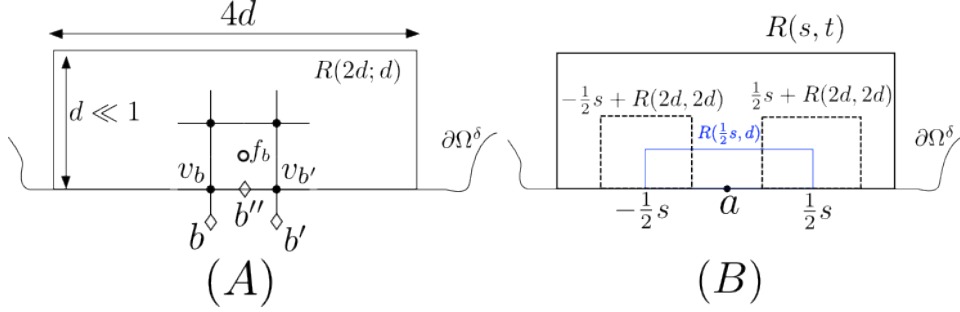


FIGURE 3. (A) Notation near b^δ used in the proof of Lemma 4.8.
 (B) Notation near a^δ used in the proof of Lemma 4.9.

Proof. Below we assume that Ω and Ω^δ are shifted and rotated so that $a^\delta = a = 0$, Ω^δ contains a discrete rectangle $R^\delta(s, t)$, and $\partial\Omega^\delta$ locally coincides with $\partial\mathbb{C}_+^\delta$. We also write $F_{1,2}^\delta$ for $F_{\varpi_{1,2}}^\delta$ and $f_{1,2}$ for $f_{\varpi_{1,2}}$. Denote

$$K^\delta := \frac{\beta_1(\delta)F_1^\delta(a^\delta)}{\beta_2(\delta)F_2^\delta(a^\delta)}.$$

Recall that both $F_j^\delta(a)$ are positive multiples of some fixed complex number $i\eta_a$ (see Proposition 3.6), thus $K^\delta > 0$. Taking a subsequence, we may assume that $K^\delta \rightarrow k < +\infty$ as $\delta \rightarrow 0$ (if $k = +\infty$, then swap F_1^δ and F_2^δ and consider the inverse ratio $(K^\delta)^{-1}$). Note that

$$\beta_{1,2}(\delta)F_{1,2}^\delta(z) \xrightarrow[\delta \rightarrow 0]{} f_{1,2}(z) = ic_{1,2}^a z^{-1} + O(1) \text{ inside } R(s, t).$$

Therefore, it is sufficient to prove that the function

$$F^\delta(z) := K^\delta \cdot \beta_2(\delta)F_2^\delta(z) - \beta_1(\delta)F_1^\delta(z),$$

defined in $R^\delta(s, t)$, converges to a limit which remains bounded near $a = 0$, since this will immediately give $kc_2^a - c_1^a = 0$ for any subsequential limit.

Note that, being a real linear combination of F_1^δ and F_2^δ , the function F^δ is s-holomorphic in the discrete rectangle $R^\delta(s, t)$ and satisfies the boundary condition (3.5) on its bottom side, including the point a^δ , where $F^\delta(a^\delta) = 0$. Thus, (4.1) allows one to define a discrete integral $H^\delta := \text{Im} \int^\delta (F^\delta(z))^2 d^\delta z$ inside $R^\delta(s, t)$ so that $H^\delta \equiv 0$ everywhere on the bottom side of $R^\delta(s, t)$.

We claim that both functions $\beta_j(\delta)F_j^\delta$, and hence F^δ and H^δ , are uniformly bounded on the top, left and right sides of the smaller rectangle $R^\delta(\frac{1}{2}s, d)$ (see Fig. 3B for the notation), where $d \ll s$. On the top side, this follows from the uniform convergence of $\beta_j(\delta)F_j^\delta$ to a continuous limit. In order to prove a uniform bound on the left and the right sides, note that, for $u \in \pm\frac{1}{2}s + R(2d, 2d)$, the second terms in (4.8) and (4.9) can be uniformly estimated using (4.3) in the following way:

$$1 - \text{hm}_{\Omega^\delta(r')}(u, \gamma_0^\delta) \leq \text{const}(s, t, r') \cdot \text{Im}u.$$

Therefore, $(\beta_j(\delta))^2 H_j^\delta(u) = O(\text{Im}u)$ in neighborhoods of the left and the right sides of $R^\delta(\frac{1}{2}s, d)$. Hence, $\beta_j(\delta)F_j^\delta = O(1)$ on those sides due to [CS12, Theorem 3.12].

Thus, H^δ is uniformly bounded on the top, left and right sides of the rectangle $R^\delta(\frac{1}{2}s, d)$ and vanishes on its bottom side. Using super-/sub-harmonicity of H^δ

on faces/vertices, and uniform estimates (4.3), one easily deduces from here that $H^\delta(u) = O(\text{Im}u)$ everywhere in $R^\delta(\frac{1}{4}s, d)$. Applying [CS12, Theorem 3.12] once again, we conclude that $F^\delta = O(1)$ in $R^\delta(\frac{1}{4}s, d)$. Hence, $kc_2^a - c_1^a = 0$. \square

Proof of Theorem 3.16. One has

$$\frac{F_{\varpi_1}(b^\delta)F_{\varpi_2}(a^\delta)}{F_{\varpi_1}(a^\delta)F_{\varpi_2}(b^\delta)} = \frac{\beta_2(\delta)F_{\varpi_2}(a^\delta)}{\beta_1(\delta)F_{\varpi_1}(a^\delta)} \cdot \frac{\beta_1(\delta)F_{\varpi_1}(b^\delta)}{\beta_2(\delta)F_{\varpi_2}(b^\delta)} \rightarrow \frac{(c_1^a)^{-1}f_{\varpi_1}^\Omega(a, b)}{(c_2^a)^{-1}f_{\varpi_2}^\Omega(a, b)},$$

where we have applied Lemmas 4.9 and 4.8 to the first and the second factors. \square

Proof of Remark 3.19(ii). Let $r > 0$ be chosen small enough. As above, it is sufficient to prove (3.13) for any subsequence $\delta = \delta_k \rightarrow 0$ such that $\beta(\delta)F_0^\delta$ converge to some nontrivial continuous limit. But this immediately follows from Lemma 4.8 applied to both boundary points $b^\delta \rightarrow b$ and $d^\delta \rightarrow d$, since the properly normalized observables $\beta(\delta)F_0^\delta(a^\delta, \cdot)$ converge to $f_0^\Omega(a, \cdot)$ at both b and d . \square

5. MULTIPLE BOUNDARY CHANGE OPERATORS

In this section, we follow [Hon10] to extend the definition of the spinor observables to the case of *multiple* marked points on the boundary. We show that these observables (we call them *multi-source* ones) are still s-holomorphic. Moreover, by analysing boundary value problems they solve, we prove *recurrence relations* that eventually allow one to express all these observables in terms of the basic ones introduced in Section 3.

The reason to introduce the multi-source observables is revealed in Propositions 5.4 and 5.6, where we establish their relation to spin correlations and partition functions, respectively. The latter is especially important in view of two applications. First, it allows one to prove the discrete martingale property of those observables with respect to interfaces growing in multiply connected domains, leading to a description of scaling limits thereof [Izy11]. Second, in the case of $2n$ microscopic holes carrying one boundary change operator each, the Kramers-Wannier duality relates the corresponding partition function to the $2n$ -points spin-spin correlations in the critical Ising model with free boundary conditions. This can be used to prove the conformal covariance of their scaling limits, which would complement the results of [CHI12] and this paper.

Consider a domain Ω^δ with $2n+1$ marked points $a := a_0, a_1, \dots, a_{2n} \in \partial\Omega^\delta$ and an inner edge z . Each configuration $S \in \text{Conf}_{a_0, a_1, \dots, a_{2n}, z}$ can be decomposed into a collection of (mutually disjoint, non-self-intersecting) loops and $n+1$ curves connecting a_k 's and z in some manner. In order to define the complex phase of S , we draw n artificial arcs ν_1, \dots, ν_n connecting $\varpi(a_1)$ to $\varpi(a_2)$, \dots , $\varpi(a_{2n-1})$ to $\varpi(a_{2n})$, respectively, and fix the way how they lift to the double cover $\tilde{\Omega}$. Adding these arcs to a configuration promotes it to a collection of loops and a single curve γ running from $\varpi(a_0)$ to $\varpi(z)$. As we admit intersections of the artificial arcs with curves constituting S , this time the loops can be self-intersecting, see Figure 4.

Definition 5.1. We define the complex phase $W_\varpi(z, S)$ to be $e^{-\frac{i}{2}\text{w}(\gamma)}(-1)^{I(S)}s(z, \gamma)$ times $(-1)^{I(S)}$, where $I(S)$ is the number of loops in $S \cup \nu_1 \cup \dots \cup \nu_n$ that have zero winding modulo 4π , and other factors are as in Definition 3.1. Further, we define

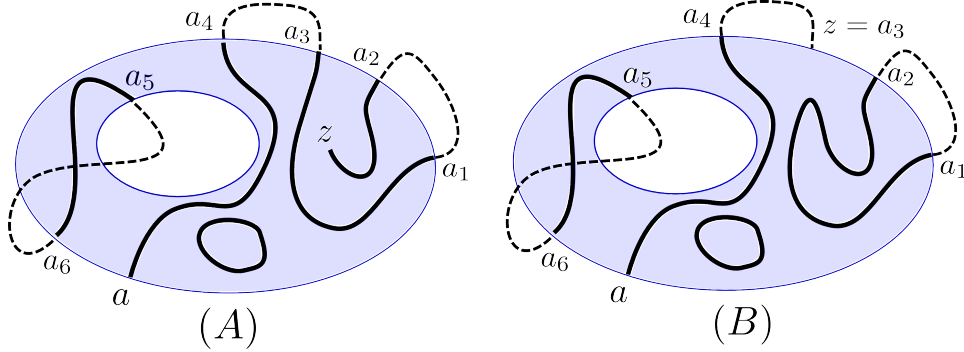


FIGURE 4. (A) A doubly connected domain Ω^δ and a configuration $S \in \text{Conf}_{a_0, a_1, \dots, a_6; z}(\Omega^\delta)$. Adding artificial arcs (dashed lines), we promote S to a collection of loops and a simple path γ running from $\varpi(a_0)$ to $\varpi(z)$. The loop containing a_5 and a_6 has winding 0, thus contributing to $I(S)$. (B) A configuration contributing to $f(a_0, a_1, a_2, a_3, a_4, a_5, a_6; a_3)$. It also contributes the same value (up to a complex factor $e^{-\frac{i}{2}w(\nu_2: a_4 \rightarrow a_3)}$) to $f(a_0, a_1, a_2, a_5, a_6; a_4)$.

the multi-source observable by the formulae

$$F_\varpi(a_0, a_1, \dots, a_{2n}; z) := i\eta_a \cdot \sum_{S \in \text{Conf}_{a, a_1, \dots, a_{2n}, z}(\Omega^\delta)} W_\varpi(z, S) x^{|S|}. \quad (5.1)$$

Remark 5.2. In fact, the only data we use concerning each of the artificial arcs ν_s is its winding $w(\nu_s)$ modulo 4π and the way how it lifts to $\tilde{\Omega}$. The reader may check that altering the choice of $\{\nu_s\}$ can only result in a sign change of the observable.

The following straightforward generalization of Theorem 3.3 holds true:

Proposition 5.3. *The observables (5.1) are s -holomorphic spinors satisfying the boundary condition (3.5) everywhere on $\partial\tilde{\Omega}^\delta \setminus \{a_0, \dots, a_{2n}, a_0^*, \dots, a_{2n}^*\}$.*

Proof. The proof is essentially the same as the one of Theorem 3.3. The only difference is that now the bijection Π between the sets of configurations $\text{Conf}_{a, a_1, \dots, a_{2n}, z'}$ and $\text{Conf}_{a, a_1, \dots, a_{2n}, z''}$ can create or destroy a loop which is self-intersecting. If such a loop has winding 2π modulo 4π , then there is no difference with the case of a simple loop. If it has winding 0 modulo 4π , then its contribution to $e^{-\frac{i}{2}w(\gamma)}$ after it becomes a part of γ is *minus* that of a simple loop, which is compensated by the simultaneous change of $I(S)$ by one. The boundary conditions follow in the same way as before. \square

We now turn to the relation of the multi-source observables with spin correlations and partition functions. Let $2n+2$ distinct boundary points $a = a_0, a_1, \dots, a_{2n}, a_{2n+1} = b$ be chosen on the *outer* boundary of $\tilde{\Omega}^\delta$, and let ϖ be a double cover that branches around boundary components $\gamma_1, \dots, \gamma_m$ and does not branch around the others. Write also $\sigma(\Gamma) := \sigma(\gamma_1) \dots \sigma(\gamma_m)$, with $\sigma(\Gamma) = 1$ if $m = 0$.

Proposition 5.4. *We have*

$$F_\varpi(a, A; b) = \pm \eta_b Z_{a, A, b} \mathbb{E}_{a, A, b}[\sigma(\Gamma)], \quad (5.2)$$

where the subscripts refer to the model with boundary change operators at all the marked points a, a_1, \dots, a_{2n}, b .

Proof. The proof follows from definition of $F_\varpi(a, A; \cdot)$ similarly to (3.6), so we leave it to the reader. \square

Remark 5.5. Let us describe one way to fix the sign in (5.2). Suppose that, starting at a and tracing the outer boundary of $\tilde{\Omega}^\delta$ in the counterclockwise direction, we find the marked points a, a_1, \dots, a_{2n}, b in that order. Let the artificial arcs ν_s run outside the domain, following the boundary arcs (a_{2s-1}, a_{2s}) . Then, assuming that the arcs $(\varpi(a_{2s-1}), \varpi(a_{2s}))$ carry “+” boundary conditions, one can replace $\pm\eta_b$ in the right-hand side of (5.2) by $-\eta_a e^{-\frac{i}{2}w_{ab}}$, where w_{ab} is the winding of the counterclockwise boundary arc $(\varpi(a), \varpi(b))$, cf. Remark 3.7.

For the next proposition, we allow the marked points $a = a_0, \dots, a_{2n+1} = b$ to be *anywhere* on the boundary. Denote by ϖ_Z the double cover that branches around all boundary components of Ω^δ carrying an *odd* number of marked points and does not branch around the others.

Proposition 5.6. *Let Ω^δ be a discrete domain. One has*

$$F_{\varpi_Z}(a, A; b) = \pm\eta_b \cdot Z_{a,A,b}.$$

Proof. Any configuration $S \in \text{Conf}_{a,A,b}$ contributes the same value $\pm\eta_b x^{|S|}$ to both sides of the equation, so it is sufficient to check that $W(b, S)$ does not depend on S . It is convenient to add one more artificial arc ν_0 connecting b to a , and to think that the arcs ν_0, \dots, ν_n are actually drawn on Ω^δ as simple, mutually non-intersecting curves, cf. Remark 5.2.

By definition, an inner boundary component has an odd number of artificial arcs ν_s issuing therefrom if and only if ϖ_Z branches around that component. Hence, cutting along, say, the right-hand side of each ν_s , we get a sheet of $\tilde{\Omega}^\delta$, and $\tilde{\Omega}^\delta$ can be obtained from two copies of that sheet by gluing along the cuts, in such a way that crossing any cut would change the sheet.

We now want to show that $W(b, S) = e^{-\frac{i}{2}w(\gamma)}(-1)^{l(S)+I(S)}s(b, \gamma)$ does not depend on S . Adding ν_0 to $S \cup \nu_1 \cup \dots \cup \nu_n$, we end up with a collection of loops; denote $\gamma_0 := \gamma \cup \nu_0$. Observe that a loop has winding 0 modulo 2π (and hence contributes to $I(S)$) if and only if it has an odd number of self-intersections, and it contributes to $l(S)$ if and only if it intersects an odd number of cross-cuts. So, we can write $l(S) + I(S) = \sum_{\text{loops } \gamma_\alpha \neq \gamma_0} r(\gamma_\alpha) \bmod 2$, where $r(\gamma_\alpha)$ is the number of intersections of $\gamma_\alpha \setminus \{\nu_1, \dots, \nu_n\}$ with *other* loops (self-intersections contribute to both $l(S)$ and $I(S)$). Those intersections can only occur between S and artificial arcs, and each intersection contributes at most once to the sum.

Further, observe that $e^{-\frac{i}{2}w(\gamma)} = e^{\frac{i}{2}w(\nu_0)}e^{-\frac{i}{2}w(\gamma_0)}$. Just as above, $-e^{-\frac{i}{2}w(\gamma_0)}$ counts the number of self-intersections of γ_0 , and $s(b, \gamma)$ describes the number of intersections of γ with the other cross-cuts. Therefore,

$$W(S, b) = -e^{\frac{i}{2}w(\nu_0)}(-1)^{r(\gamma_0)+\sum_{\text{loops } \gamma_\alpha \neq \gamma_0} r(\gamma_\alpha)}.$$

Since the exponent is the total number of intersections between the loops (not counting self-intersections), it is always even. \square

In accordance with our convention “mod 2” in (2.2), Definition 5.1 also gives the values of F_ϖ at marked boundary points. Denote $A := \{a_1, \dots, a_{2n}\}$.

Proposition 5.7. *The identity*

$$F_{\varpi}(a_0, A; z) = \sum_{k=0}^{2n} \frac{F_{\varpi}(a_0, A; a_k)}{F_{\varpi}(a_k; a_k)} F_{\varpi}(a_k; z) \quad (5.3)$$

is fulfilled for any $z \in \tilde{\Omega}^{\delta}$.

Proof. Both sides of (5.3) are discrete s-holomorphic spinors (defined on the same double cover $\tilde{\Omega}^{\delta}$) satisfying the boundary condition (3.5) everywhere on $\partial\tilde{\Omega}^{\delta}$ except the marked points a_0, \dots, a_{2n} . Moreover, for any $k = 0, \dots, 2n$, there is only one term in the sum (5.3) that fails to satisfy (3.5) at a_k . However, its value at a_k coincides with the left-hand side value $F_{\varpi}(a_0, A; a_k)$. Hence, these two s-holomorphic spinors are equal to each other due to Remark 4.2, since their difference satisfies the boundary condition (3.5) *everywhere* on $\partial\tilde{\Omega}^{\delta}$. \square

Note that configurations contributing to $F_{\varpi}(a_0, A; a_k)$ actually have $2n$ boundary points instead of $2n+2$. Thus, the right-hand side of (5.3) can be expressed in terms of the similar observables with smaller number of marked points, and, recursively, in terms of the basic observables $F_{\varpi}(a_k; \cdot)$. In order to do this in a convenient way, we need an additional notation. Recall that, for each a_k , we fix the complex number $\eta_k := \eta_{a_k}$ according to (3.1). Then, we define the *real* antisymmetric $(2n+1) \times (2n+1)$ matrix $G_{\varpi} = [(G_{\varpi})_{j,k}]$ by setting, for $0 \leq j < k \leq 2n$,

$$(G_{\varpi})_{j,k} := \frac{F_{\varpi}(a_j; a_k)}{iF_{\varpi}(a_k; a_k)} = -\frac{F_{\varpi}(a_k; a_j)}{iF_{\varpi}(a_j; a_j)} =: -(G_{\varpi})_{k,j}, \quad (5.4)$$

where we have used that $(i\eta_k)^2 e^{-i\mathbf{w}(\gamma_{kj})} = \eta_j^2$ for any curve $\gamma_{kj} : \varpi(a_k) \rightarrow \varpi(a_j)$ running in Ω^{δ} . Further, let $G_{\varpi}[k_1, \dots, k_s]$ denote the sub-matrix of G_{ϖ} obtained by *removing* rows and columns with indices k_1, \dots, k_s .

Proposition 5.8. *We have*

$$F_{\varpi}(a_0, A; z) = \pm \sum_{k=0}^{2n} (-1)^k \text{Pf } G_{\varpi}[k] \cdot F_{\varpi}(a_k; z), \quad (5.5)$$

with the sign depending on the choices made for η_k and ν_s . In particular, the sign is “plus” with conventions described in Remark 5.9 below.

Remark 5.9. (i) The sign in the left-hand side of (5.5) depends on the choice of artificial arcs ν_s , while for the sum in the right-hand side it depends on the sheets of a_k and the signs of η_k , $k = 1, \dots, 2n$. We will assume that each ν_s lifts to a path from a_{2s-1} to a_{2s} on $\tilde{\Omega}^{\delta}$, and that the signs of η_k are chosen so that

$$\exp \left[-\frac{i}{2} \mathbf{w}(\nu_s : a_{2s-1} \rightarrow a_{2s}) \right] = \frac{\eta_{2s}}{i\eta_{2s-1}}. \quad (5.6)$$

(ii) One could also write (5.5) as the Pfaffian of a $(2n+2) \times (2n+2)$ matrix, obtained from G_{ϖ} by adding the last column with entries $F_{\varpi}(a_k; z)$ and a corresponding row.

Proof. We prove the claim by induction in n , starting with the trivial case $n = 0$ and using (5.3). For $k = 1, \dots, 2n$, let $k' := k + 1$, if k is odd, and $k' := k - 1$, if k is even. By definition, any configuration contributing to $F_{\varpi}(a_0, A; a_k)$ contains a

curve running from a_0 to $a_{k'}$, appended with an artificial arc connecting $a_{k'}$ to a_k . Removing that arc, and taking into account (5.6), we get the following identity:

$$F_{\varpi}(a_0, A; a_k) = (-1)^k \frac{\eta_k}{i\eta_{k'}} F_{\varpi}(a_0, A[k, k']; a_{k'}),$$

where $A[k, k'] := A \setminus \{a_k, a_{k'}\}$. Thus, using the induction hypothesis and observing that $\eta_k^{-1} F(a_k, a_k)$ does not depend on k , we get (for $k = 1, \dots, 2n$)

$$\begin{aligned} \frac{F_{\varpi}(a_0, A; a_k)}{F_{\varpi}(a_k; a_k)} &= (-1)^k \frac{\eta_k}{i\eta_{k'}} \sum_{0 \leq j \neq k, k' \leq 2n} (-1)^j \text{Pf } G_{\varpi}[j, k, k'] \frac{F_{\varpi}(a_j; a_{k'})}{F_{\varpi}(a_k; a_k)} \\ &= (-1)^k \sum_{0 \leq j \neq k, k' \leq 2n} (-1)^j \text{Pf } G_{\varpi}[j, k, k'] \frac{F_{\varpi}(a_j; a_{k'})}{iF_{\varpi}(a_{k'}; a_{k'})} \\ &= (-1)^k \sum_{0 \leq j \neq k, k' \leq 2n} (-1)^{j+1[j > k']} \text{Pf } G_{\varpi}[j, k, k'] \cdot (G_{\varpi})_{j, k'}. \end{aligned}$$

Due to the standard recursive formula for Pfaffians applied to the matrix $G_{\varpi}[k]$, this can be written as

$$\frac{F_{\varpi}(a_0, A; a_k)}{F_{\varpi}(a_k; a_k)} = (-1)^k \text{Pf } G_{\varpi}[k], \quad k = 1, \dots, 2n. \quad (5.7)$$

Similarly,

$$F_{\varpi}(a_0, A; a_0) = \frac{i\eta_0}{\eta_1} F_{\varpi}(a_2, A[1, 2]; a_1) = \frac{i\eta_0}{\eta_1} \sum_{k=3}^{2n} \frac{F_{\varpi}(a_2, A[1, 2]; a_k)}{F_{\varpi}(a_k; a_k)} F_{\varpi}(a_k; a_1).$$

Applying (5.7) and using $\eta_0^{-1} F_{\varpi}(a_0; a_0) = \eta_1^{-1} F_{\varpi}(a_1; a_1)$, one arrives at

$$\frac{F_{\varpi}(a_0, A; a_0)}{F_{\varpi}(a_0; a_0)} = - \sum_{k=3}^{2n} (-1)^k \text{Pf } G_{\varpi}[0, 1, k] \cdot (G_{\varpi})_{k, 1} = \text{Pf } G_{\varpi}[0].$$

Plugging this and (5.7) into (5.3), we obtain (5.5). \square

Corollary 5.10. *With the notation as above and conventions of Remark 5.5, let Ω^{δ} approximate Ω as $\delta \rightarrow 0$, regularly at all marked points. Then,*

$$\frac{\mathbb{E}_{a_0^{\delta} \dots a_{2n+1}^{\delta}}[\sigma(\Gamma)]}{\mathbb{E}_{+}[\sigma(\Gamma)]} \rightarrow \frac{\text{Pf}[\zeta_{a_j a_k}^{-1} \vartheta_{a_j a_k}^{\Omega}(\gamma_1, \dots, \gamma_m)]_{0 \leq j < k \leq 2n+1}}{\text{Pf}[\zeta_{a_j a_k}^{-1}]_{0 \leq j < k \leq 2n+1}}, \quad (5.8)$$

where the conformal invariants ϑ_{ab}^{Ω} are given by (3.12), and $\zeta_{ab} = \zeta_{ab}^{\Omega} := |f_0^{\Omega}(a, b)|^{-1}$.

Proof. Below we omit δ for the shortness. Using the relations (5.2) and (3.7) for ϖ as in Proposition 5.4 and for the trivial cover, and then (5.5), we get

$$\begin{aligned} \frac{\mathbb{E}_{a_0 \dots a_{2n+1}}[\sigma(\Gamma)]}{\mathbb{E}_{+}[\sigma(\Gamma)]} &= \frac{F_{\varpi}(a_0, A; a_{2n+1}) \cdot F_0(a_{2n+1}, a_{2n+1})}{F_0(a_0, A; a_{2n+1}) \cdot F_{\varpi}(a_{2n+1}, a_{2n+1})} = \\ &= \frac{\sum_{k=0}^{2n} (-1)^k \text{Pf } G_{\varpi}[k] \cdot F_{\varpi}(a_k; a_{2n+1}) \cdot (iF_{\varpi}(a_{2n+1}; a_{2n+1}))^{-1}}{\sum_{k=0}^{2n} (-1)^k \text{Pf } G_0[k] \cdot F_0(a_k; a_{2n+1}) \cdot (iF_0(a_{2n+1}; a_{2n+1}))^{-1}} \\ &= \frac{\sum_{k=0}^{2n} (-1)^k \text{Pf } G_{\varpi}[k] \cdot (G_{\varpi})_{k, 2n+1}}{\sum_{k=0}^{2n} (-1)^k \text{Pf } G_0[k] \cdot (G_0)_{k, 2n+1}} = \frac{\text{Pf}[(G_{\varpi})_{j, k}]_{0 \leq j < k \leq 2n+1}}{\text{Pf}[(G_0)_{j, k}]_{0 \leq j < k \leq 2n+1}}, \end{aligned}$$

where G_ϖ and G_0 are given by (5.4) for corresponding double covers. This can be further rewritten as

$$\frac{\mathbb{E}_{a_0 \dots a_{2n+1}}[\sigma(\Gamma)]}{\mathbb{E}_+[\sigma(\Gamma)]} = \frac{\text{Pf} \left[\frac{(G_\varpi)_{j,k}}{(G_0)_{j,k}} \cdot \frac{(G_0)_{j,k}}{(G_0)_{0,2n+1}} \right]_{0 \leq j < k \leq 2n+1}}{\text{Pf} \left[\frac{(G_0)_{j,k}}{(G_0)_{0,2n+1}} \right]_{0 \leq j < k \leq 2n+1}}.$$

Note that definition (5.4) and convergence (3.11) yield

$$\frac{(G_\varpi)_{j,k}}{(G_0)_{j,k}} = \frac{F_\varpi(a_j; a_k) F_0(a_j; a_j)}{F_\varpi(a_j; a_j) F_0(a_j; a_k)} \rightarrow \frac{f_\varpi^\Omega(a_j, a_k)}{f_0^\Omega(a_j, a_k)} = \vartheta_{a_j a_k}^\Omega(\gamma_1, \dots, \gamma_m)$$

as $\delta \rightarrow 0$. Similarly, (5.4) and (3.13) give

$$\frac{(G_0)_{j,k}}{(G_0)_{0,2n+1}} = \frac{|F_0(a_j; a_k)|}{|F_0(a_0; a_{2n+1})|} \rightarrow \frac{|f_0^\Omega(a_j, a_k)|}{|f_0^\Omega(a_0, a_{2n+1})|} = \frac{\zeta_{a_0 a_{2n+1}}}{\zeta_{a_j a_k}},$$

and the factors $\zeta_{a_0 a_{2n+1}}$ cancel out in the ratio of Pfaffians. \square

Remark 5.11. (i) One may assume that $f_0^\Omega(a, b)$ is normalized by (3.8); in this case one has a conformal covariance rule $\zeta_{a,b}^\Omega = \zeta_{\phi(a), \phi(b)}^{\phi(\Omega)} |\phi'(a)|^{1/2} |\phi'(b)|^{1/2}$. The linearity of Pfaffians implies that the normalization is not important, and that the ratio (5.8) is conformally invariant.

(ii) Clearly, f_0^Ω does not change if one adds boundary singletons inside the domain. In particular, if $\Omega = \mathbb{C}_+ \setminus \{w_1, \dots, w_s\}$ and $a, b \in \mathbb{R}$, then one has $\zeta_{a,b}^\Omega = \sqrt{\pi} |b - a|$ (with the normalization given by (3.8)).

(iii) The formula (5.8) was predicted by means of Conformal Field Theory, see [BG93, equation (17)].

6. EXPLICIT COMPUTATIONS IN THE HALF-PLANE

In this section we explicitly compute the holomorphic spinors $f_\varpi^{\mathbb{C}_+ \setminus \{w_1, \dots, w_m\}}$. By (3.12), this immediately gives us the quantities

$$\vartheta(w_1, \dots, w_m) := \vartheta_{\infty, 0}^{\mathbb{C}_+ \setminus \{w_1, \dots, w_m\}}(w_1, \dots, w_m) \quad (6.1)$$

and, by conformal invariance of ϑ 's, all the limits

$$\lim_{\delta \rightarrow 0} \frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(w_1^\delta) \dots \sigma(w_m^\delta)]}{\mathbb{E}_+[\sigma(w_1^\delta) \dots \sigma(w_m^\delta)]} = \vartheta(\phi(w_1), \dots, \phi(w_m)),$$

for discrete domains $(\Omega^\delta; a^\delta, b^\delta)$ approximating an arbitrary *simply connected* domain $(\Omega; a, b)$ and inner *faces* w_j^δ tending to $w_j \in \Omega$, where $\phi : \Omega \rightarrow \mathbb{C}_+$ is a conformal map such that $\phi(a) = \infty$ and $\phi(b) = 0$. Due to the Pfaffian formula (5.8), this result easily extends to the case of $2n+2$ marked boundary points.

It is convenient to choose $a = \infty$. Specializing the boundary value problem (a $^\circ$)–(d $^\circ$) and (3.8) to the case $\Omega = \mathbb{C}_+$, $a = 0$ and using the conformal map $z \mapsto -z^{-1}$, we get the following conditions for $f := f_\varpi^{\mathbb{C}_+ \setminus \{w_1, \dots, w_m\}}(\infty, \cdot)$:

- (a $^\circ$) f is a spinor in $\mathbb{C}_+ \setminus \{w_1, \dots, w_m\}$ branching around each of w_j ;
- (b $^\circ$) $f(\zeta) \in \mathbb{R}$ for any $\zeta \in \mathbb{R}$;
- (c $^\circ$) $(f(z))^2 = O(|z - w_j|^{-1})$ as $z \rightarrow w_j$ and $\text{res}_{z=w_j}(f(z))^2 \in i\mathbb{R}_+$ for all j ;
- (d $^\circ$) $f(z) = 1 + O(z^{-1})$ as $z \rightarrow \infty$.

Note that for $m = 0$ (i.e., the trivial cover), we have an obvious solution $f_0 \equiv 1$.

In order to find f , we introduce an auxiliary spinor

$$f_{w_1, \dots, w_m}(z) := \mathcal{B}_{w_1}(z) \cdots \mathcal{B}_{w_m}(z), \quad \mathcal{B}_w(z) := \frac{z - \operatorname{Re} w}{[(z - \bar{w})(z - w)]^{\frac{1}{2}}}.$$

Note that it satisfies (a^o), (b^o) and (d^o). Moreover, it has real zeros at $t_j = \operatorname{Re} w_j$; thus, (a^o), (b^o) and (d^o) also hold for the product $f_{w_1, \dots, w_m}(z) \cdot g(z)$, where $g(z)$ is any function of the form

$$g(z) \equiv 1 + \sum_{j=1}^m \frac{\lambda_j}{t_j - z}, \quad \lambda_k \in \mathbb{R} \quad (6.2)$$

(and if some t_j coincide, we can add higher-order poles, so that $g(z)$ is always a linear combination of m linearly independent functions). We are looking for parameters λ_j such that $f_{w_1, \dots, w_m}(z) \cdot g(z)$ satisfy (c^o) as well. Denote

$$R_k := [-2i \operatorname{res}_{z=w_k} (f_{w_1, \dots, w_m}(z))^2]^{\frac{1}{2}}.$$

Then, the condition (c^o) for $f_{w_1, \dots, w_m}(z) \cdot g(z)$ can be restated as

$$\operatorname{Im}[R_k \cdot g(w_k)] = \operatorname{Im} R_k + \sum_{j=1}^m \lambda_j \cdot \operatorname{Im} \frac{R_k}{t_j - w_k} = 0 \quad \text{for all } k = 1, \dots, m. \quad (6.3)$$

Note that (6.3) is an $m \times m$ linear system in λ_j . We argue that this system is always non-degenerate. Indeed, if λ_j^0 is a solution to the corresponding *homogeneous* system, then $f(z) := [\sum_{j=1}^m \lambda_j^0 / (t_j - z)] \cdot f_{w_1, \dots, w_m}(z)$ is a spinor satisfying (a^o)–(c^o) and such that $f(z) = O(z^{-1})$ as $z \rightarrow \infty$. But it follows from the proof of Lemma 3.10 that any such spinor is identically zero (after mapping to a bounded domain, the condition at infinity yields boundedness near a). Thus, $\lambda_j^0 \equiv 0$.

Taking into account that one has explicitly

$$R_k = (\operatorname{Im} w_k)^{\frac{1}{2}} \cdot \prod_{j \neq k} \mathcal{B}_{w_j}(w_k) = (\operatorname{Im} w_k)^{\frac{1}{2}} \cdot \prod_{j \neq k} \frac{(w_k - \operatorname{Re} w_j)}{[(w_k - \bar{w}_j)(w_k - w_j)]^{\frac{1}{2}}}.$$

and solving the linear system (6.3), one obtains λ_j as ratios of certain explicit $m \times m$ determinants (depending of w_j 's), and then the ratio of spin correlations (6.1) is given by

$$\vartheta(w_1, \dots, w_m) = f(0) = g(0) \cdot \prod_{j=1}^m \mathcal{B}_{w_j}(0) = \left[1 + \sum_{j=1}^m \frac{\lambda_j}{\operatorname{Re} w_j} \right] \cdot \prod_{j=1}^m \frac{\operatorname{Re} w_j}{|w_j|}.$$

Here we have used conventions of Remark 3.7 to determine the sheet of the double cover of $\mathbb{C}_+ \setminus \{w_1, \dots, w_n\}$ (that is, signs of the square roots): in order to get the value $f(0)$, one starts with the value $f(\infty) = +1$ and continuously moves the boundary point to 0 along the *counterclockwise* boundary arc $(-\infty, 0)$, thus arriving to $\mathcal{B}_{w_j}(0) = (-\operatorname{Re} w_j)/(-|w_j|)$. The other way to fix the sign is given by taking all w_j close to the boundary arc $(0; +\infty)$: this should yield a positive correlation as that arc carries “+” boundary conditions.

Remark 6.1. In particular, for a *single* point $w \in \mathbb{C}$ (i.e., for the case $m = 1$) one has $t = \operatorname{Re} w$, $R = (\operatorname{Im} w)^{1/2} \in \mathbb{R}$, thus $\lambda = 0$ and

$$\vartheta(w) = \frac{\operatorname{Re} w}{|w|} = \cos[\pi \operatorname{hm}_{\mathbb{C}_+}(w, \mathbb{R}_-)].$$

Since harmonic measure is conformally invariant, this implies

$$\frac{\mathbb{E}_{a^\delta b^\delta}[\sigma(w^\delta)]}{\mathbb{E}_+[\sigma(w^\delta)]} \rightarrow \cos[\pi \text{hm}_\Omega(w, (ab))],$$

if $(\Omega^\delta; a^\delta, b^\delta)$ approximate $(\Omega; a, b)$ and faces w^δ tend to an inner point $w \in \Omega$.

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